On the Symmetric Tensor Operators of the Unitary Groups

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The algebraic expressions for the matrix elements of symmetric tensor operators (the powers of infinitesimal operators) of the unitary groups in the Gel'fand basis have been studied. The expressions for the isoscalar factors of the related Clebsch–Gordan coefficients, one of the two representations to be coupled being symmetric, as well as the elements of a special recoupling matrix have been found. The supplementary symmetry properties of the isoscalar factors corresponding to the Regge symmetries of the Wigner and $6j$ coefficients of SU_2 have been examined.

1. INTRODUCTION

The mathematical apparatus of the irreducible tensor operators of the unitary groups[1] is a very important generalization of the theory of angular momentum of contemporary theoretical physics. The main problem of this apparatus is to obtain the algebraic expressions for Clebsch–Gordan (CG) coefficients, recoupling matrices and matrix elements of irreducible tensor operators. It is useful at first to solve the simpler special problems, for example, to consider the matrix elements of the extremal tensor operators of the unitary groups.[2]

An interesting and more difficult problem is to obtain the expressions for the matrix elements of the symmetric tensor operators, which enables one to find the expressions for the general tensor operators. The main aim of this paper is to consider these symmetric operators. We take the product of powers of commuting infinitesimal operators (generators) of U_{n+1} of the type E_{in+1} $(i = 1, 2, \ldots, n)$, as a realization of the symmetric tensor operator of U_n. The matrix elements of such an operator can be expressed as a product of the reduced matrix elements and isoscalar factors (i.f.) of CG coefficients with one of the two representations symmetric. The combinatorial-graphical techniques for calculating such a special i.f. has been found by one[3] of the authors by the use of the Young operators of the symmetric groups as projection operators. The expressions of Ref. 3 are not optimal ones with respect to the number of terms in the sum, because the summation is taking place over the permutations of labeled squares of the Young tableau. Here we are going to obtain the corresponding expressions with a greatly reduced number of terms in the sums, the summation taking place over integers only. Furthermore, a rather simple recurrence method is described for obtaining the expressions under consideration. The technique resembles that used by one of the authors[4] in the case of SU_3. The two classes of expressions are given and their symmetry properties, partly given in Ref. 3, discussed.

In Appendix A it is shown how the corresponding formulas can be obtained from the results of Ref. 3.

In Appendix B there is given a useful relation between the isoscalar factors of the Gel'fand basis and the elements of the recoupling matrix described in Ref. 5.

The results of this paper can be used for obtaining the general expressions for the CG coefficients of U_n. For this purpose one can use the projection operators in the form of polynomials of infinitesimal operators[6] as has been done in the case of SU_3.[7] Alternatively, one can use the recurrence relations obtained by forming the general tensor operators from the symmetric ones. The normalization procedure in the second case can be carried out using the relation of Ref. 5.

2. THE MATRIX ELEMENTS OF POWERS OF INFINITESIMAL OPERATORS

At first we obtain the expression for the i.f. of a special kind

$$\begin{bmatrix} [m]_n & p & [m']_n \\ [m]_{n-1} & 0 & [m]_{n-1} \end{bmatrix} = \Big[p! \prod_{1\le j<k\le n} (m'_{jn} - m'_{kn} - j + k)\Big]^{1/2} \times \frac{D([m]_n, [m']_n)\Gamma([m']_n, [m]_{n-1})}{\Gamma([m]_n, [m]_{n-1})}. \tag{1}$$

Here $[m]_k \equiv [m_{1k}, m_{2k}, \dots, m_{kk}]$ means the corresponding row of the Gel'fand pattern of the representation of U_n (c.f. Ref. 1), p is the single parameter of the symmetric representation. In (1), and in what follows, we use the notations

$$\Gamma([m]_n, [m]_{n-1}) = \left(\frac{\prod_{1\le i\le j\le n-1} (m_{in} - m_{jn-1} - i + j)!}{\prod_{1\le j<i\le n} (m_{jn-1} - m_{in} + i - j - 1)!}\right)^{1/2}; \tag{2}$$

$$D([m]_n, [m']_n) = \left(\frac{\prod_{1\le i<j\le n} (m_{in} - m'_{jn} - i + j - 1)!}{\prod_{1\le j\le i\le n} (m'_{jn} - m_{in} + i - j)!}\right)^{1/2}. \tag{3}$$

The dependence of the i.f. of Eq. (1) on the parameters of the representations of the subgroup U_{n-1} is confined to Γ, this dependence being deduced by factorizing the simpler i.f.[8] The remaining part of the expression on the right-hand side of Eq. (1) is a normalization factor. This factor can be deduced by equating the particular case of Eq. (1) ($[m]_{n-1} = [m]_n$, $m_{nn} = 0$) to the one calculated with the help of projection operators (Refs 6 and 7). It must be noted that we use the general weight lowering operators of the form

$$F_- \begin{matrix}[h]_n \\ [h']_{n-1}\end{matrix} = \prod_{i=1}^{n-1}\left(\frac{1}{(h_i - h'_i)} \prod_{s=i+1}^{n-1} \frac{(h_i - h'_s - i + s)!}{(h'_i - h'_s - i + s)!} \times \prod_{s=i+1}^{n} \frac{(h'_i - h_s - i + s - 1)!}{(h_i - h_s - i + s - 1)!}\right)^{1/2} \times P^{n-1}_{max}[h']_{n-1} \prod_{i=1}^{n-1} E_{ni}^{\,h_i - h'_i}, \tag{4}$$

rather than that of Ref. 9. Here $P^{n-1}_{max}[h']_{n-1}$ is a projection operator of maximal weight as defined in Ref. 6.

The easiest way to obtain (1) is to use the results of Ref. 3 [these last ones being contained in Eq. (A2) of Appendix A]. The sum $F_{p_1}(x_i - y_k)$ in this case reduces to one term equal to 1.

Let us now consider the calculation of the matrix elements of powers of generators. The simplest cases of them are obtained by factorizing the matrix elements of individual generators of the group. To these cases belong, in the first place, the matrix elements stretched with respect to the parameters of the representations of subgroups.

The simplest of powers of generators seems to be $(E_{n-1\,n})^p$. With respect to the subgroup U_{n-1}, this operator corresponds to the scalar component U_{n-2} of the symmetric tensor T^p. Hence this matrix element is proportional to the i.f. of U_{n-1} being calculated with the help of (1). The corresponding reduced matrix elements are to be obtained from the relation

$$\left\langle \begin{matrix}[m]_n \\ [m'']_{n-1}\end{matrix} \middle\| E_{n-1n}{}^{p} \middle\| \begin{matrix}[m]_n \\ [m']_{n-1}\end{matrix} \right\rangle = \begin{bmatrix} [m']_{n-1} & p & [m'']_{n-1} \\ [m']_{n-2} & 0 & [m']_{n-2} \end{bmatrix}^{-1} \times \left\langle \begin{matrix}[m]_n \\ [m'']_{n-1}\end{matrix}_{\max} \middle| F_-^{\dagger} \begin{matrix}[m'']_{n-1} \\ [m']_{n-2}\end{matrix} (E_{n-1n})^p \middle| \begin{matrix}[m]_n \\ [m']_{n-1}\end{matrix}_{\max} \right\rangle. \tag{5}$$

Here $m'_{in-2} = m'_{in-1}$ $(i \le n-2)$ is the maximal weight of U_{n-2}. The operator $P^{n-2}_{\max}[m']_{n-2}$ in F_- gives unity in this case. With the help of Eq. (2.11) of Ref. 7 we transpose E_{n-1n} with $F_-^{\dagger}$. All the powers of E_{in-1} with nonvanishing exponents acting on the maximal state of U_{n-1} give zero. For this reason, summations arising in the process of the transposition disappear and we are left with the matrix element of the operator

$$\frac{p!}{(m''_{n-1n-1} - m'_{n-1n-1})!} \prod_{i=1}^{n-1} E_{in}^{\,m''_{in-1} - m'_{in-1}},$$

which is stretched in this case.

The operations described above give the following expression for the reduced matrix element under consideration:

$$\left\langle \begin{matrix}[m]_n \\ [m']_{n-1}\end{matrix} \middle\| E_{n-1n}{}^{p} \middle\| \begin{matrix}[m]_n \\ [m]_{n-1}\end{matrix} \right\rangle = \delta\left(\sum_{i=1}^{n-1} m'_{in-1}, p + \sum_{i=1}^{n-1} m_{in-1}\right)\Big[p! \times \prod_{1\le i<j\le n-1} (m_{in-1} - m_{jn-1} - i + j)\Big]^{1/2} \times \frac{D([m]_{n-1}, [m']_{n-1})\Gamma([m]_n, [m]_{n-1})}{\Gamma([m]_n, [m']_{n-1})}. \tag{6}$$

It is to be noted that this reduced matrix element coincides with those of the operators

$$\left(\frac{p!}{\prod_{i=1}^{n-1}\alpha_i!}\right)^{1/2}\prod_{i=1}^{n-1} E_{in}{}^{\alpha_i},\left(\sum_{i=1}^{n-1}\alpha_i = p\right), \tag{7}$$

because they form the basis of the representation p of U_{n-1}. The corresponding matrix elements themselves are obtained by multiplying the reduced matrix elements obtained above by the products of i.f.'s to be dealt with in what follows.

3. ISOSCALAR FACTOR WITH ONE OF THE REPRESENTATIONS SYMMETRIC

In generalizing and simplifying the method of Ref. 4, we take the lowest weight component T_0^p of the symmetric unit operator of U_n. Its matrix element with respect to the corresponding basis is equal to the i.f. given by Eq. (1). More general components of the same operator are

$$T_q^p = \left(\frac{(p-q)!}{p!q!}\right)^{1/2} \times [\underbrace{E_{n-1n}[E_{n-1n}[\dots[E_{n-1n}}_{q \text{ times}}T_0^p]\dots]]] = \left(\frac{(p-q)!q!}{p!}\right)^{1/2}\sum_x \frac{(-1)^x}{x!(q-x)!} E_{n-1n}{}^{q-x}\, T_0^p E_{n-1n}{}^x. \tag{8}$$

The reduced matrix element of this operator with respect to the subgroup U_{n-1} is the i.f. under consideration; it is

$$\begin{aligned}\begin{bmatrix}[m]_n & p & [m']_n\\ [m]_{n-1} & q & [m']_{n-1}\end{bmatrix} &= [(p-q)! \prod_{1\le i<j\le n-1}(m_{in-1}-m_{jn-1}-i+j) \\ &\times \prod_{1\le i<j\le n}(m'_{in}-m'_{jn}-i+j)]^{1/2} \\ &\times \frac{D([m]_n,[m']_n)\Gamma([m]_n,[m]_{n-1})}{D([m]_{n-1},[m']_{n-1})\Gamma([m']_n,[m']_{n-1})} \\ &\times \sum_{[r]_{n-1}}(-1)^{\sum_{i=1}^{n-1}(r_{in-1}-m_{in-1})} \\ &\times \prod_{1\le i<j\le n-1}(r_{in-1}-r_{jn-1}-i+j) \\ &\times D^2([m]_{n-1},[r]_{n-1})D^2([r]_{n-1},[m']_{n-1}) \\ &\times \frac{\Gamma^2([m']_n,[r]_{n-1})}{\Gamma^2([m]_n,[r]_{n-1})},\end{aligned}$$

$$p=\sum_{i=1}^{n}(m'_{in}-m_{in}), \quad q=\sum_{i=1}^{n-1}(m'_{in-1}-m_{in-1}), \tag{9}$$

$[r]_{n-1}$ being the Young scheme and the summation taking place over $n-1$ parameter r_{in-1}. When both of the two representations to be coupled are symmetric, expression (9) reduces to the CG coefficient of SU_2 [the second of Eq. (13.1) of Ref. 10]. On the other hand, when $n=3$ it turns into Eq. (3.14) of Ref. 4. Furthermore, we can limit ourselves to the case $m_{nn}=0$ which does not influence the value of the i.f., as pointed out in Ref. 1.

It is easy to see that (9), after omitting the square root, possesses the high symmetry indicated in Ref. 3. For example, it is possible to transpose the parameters m_{in-1} and m'_{i+1n} with the appearance of the phase factor $(-1)^{m_{in-1}-m'_{i+1n}}$. The other kind of Regge symmetry gives the transposition of m_{in} with m'_{in-1}, without any phase factor. For the tabulation of the symmetric part of (9), it is useful to apply the following scheme of $4n-2$ parameters:

$$m'_{1n}, \max(m'_{1n-1}, m_{1n}), \min(m'_{1n-1}, m_{1n}), \max(m'_{2n}, m_{1n-1}), \min(m'_{2n}, m_{1n-1}), \max(m'_{2n-1}, m_{2n}), \dots, m_{nn}=0 \tag{10}$$

arranged in a lexical order and using specified phase relations for the transpositions of the first Regge symmetry type.

Another symmetry property follows from the contragredience relations.[8] This procedure gives $(-1)^q$ as a phase factor, and the set of parameters (10) turns into the set obtained from this one by changing the signs and writing in inverted order, all the parameters becoming positive after adding m'_{1n}.

In this way one obtains 2^{2n-1} symmetry properties for the quantity (9). It stands to reason that not all the Regge symmetry properties[11] of quantities of SU_2 can be generalized to SU_n with $n>2$.

We observe that the symmetry property of Ref. 3 allowing one to interchange the rows in the skew scheme belongs to the substitution group symmetry[12] rather than to one of the Regge type. Equation (9) is invariant with respect to this group which is equivalent to partial hook permutations (c.f. Ref. 1).

It is to be noted that the relation between i.f.'s which couple the bases of two symmetric representations (of equal or different contragrediency) and SU_2 CG coefficients[13] follows immediately from the Regge and substitution symmetry properties.

Expression (9) does not allow one to carry out the summations even for particular cases. Thus, it is worthwhile to use other methods to obtain different expressions for the same i.f. We can obtain one such expression by the use of the operator

$$\left(\frac{p!}{(p-q)!q!}\right)^{1/2} E_{nn+1}{}^{p-q} E_{n-1n+1}{}^q = \left(\frac{p!}{(p-q)!q!}\right)^{1/2}\sum_y \frac{(-1)^{y-\alpha}}{(y-\alpha)!(q-y+\alpha)!} \times E_{nn+1}{}^y E_{n-1n}{}^q E_{nn+1}{}^{p-y} \tag{11}$$

instead of (8). After dividing its matrix element by the reduced matrix element of the operator $E_{nn+1}{}^p$ and the i.f. of U_{n-1}, one obtains

$$\begin{aligned}\begin{bmatrix}[m]_n & p & [m']_n\\ [m]_{n-1} & q & [m']_{n-1}\end{bmatrix} &= [(p-q)!]^{-1/2}[\prod_{1\le i<j\le n}(m'_{in}-m'_{jn}-i+j) \\ &\times \prod_{1\le i<j\le n-1}(m_{in-1}-m_{jn-1}-i+j)]^{1/2} \\ &\times \frac{D([m]_{n-1},[m']_{n-1})\Gamma([m']_n,[m']_{n-1})}{D([m]_n,[m']_n)\Gamma([m]_n,[m]_{n-1})} \\ &\times R\begin{pmatrix}[m]_n & [m']_n\\ [m]_{n-1} & [m']_{n-1}\end{pmatrix},\end{aligned} \tag{12}$$

where

$$R\left(\begin{matrix}[m]_n \\ [m]_{n-1}\end{matrix}\quad\begin{matrix}[m']_n \\ [m']_{n-1}\end{matrix}\right) = \sum_{[r]_n} \frac{(-1)^{y-\alpha} y!(p-y)!\Gamma^2([r]_n,[m]_{n-1})}{(y-\alpha)!(q-y+\alpha)!\Gamma^2([r]_n,[m']_{n-1})} \times \prod_{1\le i<j\le n} (r_{in}-r_{jn}-i+j) D^2([r]_n,[m']_n) \times D^2([m]_n,[r]_n), \quad y=\sum_{i=1}^{n}(m'_{in}-r_{in}). \tag{13}$$

The number of the summation parameters in (13) is n. The terms of this sum depend on $\alpha (0 \le \alpha \le p-q)$. However, the final result must be independent of this parameter. It turns out that in expression (13) the summation with respect to one of parameters r_{in} can be carried out by the use of the summation formula

$$\sum_x \frac{(-1)^x \prod_{i=1}^{a}(x+A_i)\prod_{i=1}^{b}(B_i-x)}{x!(a+b+c-x)!} = (-1)^a \delta(c,0), \tag{14}$$

a, b, c being nonnegative integers. Equation (14) can be proved by induction starting from Eq. (14.3) of Ref. 10.

In order to use this summation formula for the purpose indicated in Eq. (13), we transform the factorials depending on r_{in} (i fixed) into the quasipowers (c.f. Ref. 10), all these being brought into the numerator. The factors left in the denominator are

$$(m'_{1n}-r_{in}+i-1)!(r_{in}-m_{nn}-i+n)!.$$

It is evident that the sum in (13) in this new form has a much wider summation region, because it involves $n-1$ new regions. However, this procedure does not change the value of the sum (13), because nonvanishing terms in these new regions are compensated by a set of terms equal in absolute value and opposite in sign to the first ones. These terms can be found by renumerating the summation parameters $r_{jn}-j \longleftrightarrow r_{in}-i,\ j\ne i$ labeling the newly appearing regions.

The above mentioned summation with respect to r_{in} leads us to the expression

$$R_i\left(\begin{matrix}[m]_n \\ [m]_{n-1}\end{matrix}\quad\begin{matrix}[m']_n \\ [m']_{n-1}\end{matrix}\right) = \sum_{r_{jn}, j\ne i} (-1)^{\varphi_i} \times \prod_{\substack{1\le k<l\le n \\ k\ne 1,\, l\ne i}} (r_{kn}-r_{ln}-k+l) \times D^2_{i,0}([r]_n,[m']_n) D^2_{0,i}([m]_n,[r]_n) \frac{\Gamma^2_{i,0}([r]_n,[m]_{n-1})}{\Gamma^2_{i,0}([r]_n,[m']_{n-1})}$$

$$\varphi_i = \sum_{j=1}^{i-1}(m'_{jn-1}-m_{jn-1}+m_{jn}) + \sum_{j=i+n}^{n} m'_{jn} - \sum_{\substack{j=1\\ j\ne i}}^{n} r_{jn}. \tag{15}$$

The quantities $D_{i,0}$, $D_{0,i}$, and $\Gamma_{i,0}$ are obtained from those of Eqs. (2) and (3) by removing those factors involving parameters with subscripts i, out of $[r]_n$.

Since all R_i $(i=1,2,\ldots,n)$ in (15) are equivalent, they are connected by the elements of the substitution group of Ref. 12. R_1 and R_n are more convenient for some problems. For example, in the semistretched case $(m'_{nn}=m_{nn})$, it is useful to take R_1. On the other hand, in the case of the maximal state $(m'_{in-1}=m'_{in}, i=1,2,\ldots n-1)$, it is more convenient to use R_n.

In the case of SU_2, our expression turns into the third of Eqs. (13.1) of Ref. 10. On the other hand, in the case of the semistretched coupling of representations of SU_3 (Ref. 4) it leads us to the doubly stretched $9j$ coefficient of SU_2 given by Eq. (25.17) of Ref. 10.

It is worth noting that our expressions (13) and (15) have n regions for n summation parameters in the case of Eq. (13) and $n-1$ summation parameters in the case of Eq. (15). In the second case, one of the summation regions is free. This occurs because Eq. (15) does not possess the full Regge symmetry exhibited by Eq. (9), which has $n-1$ summation parameters as well as regions.

APPENDIX A: AN ALTERNATIVE APPROACH TO THE PROBLEM

Let $[\lambda^1], [\lambda^2], [\lambda^3], [\lambda^4]$ be the Young schemes such that $\lambda_i^1 \le \lambda_i^2 \le \lambda_i^3 \le \lambda_i^4$ $(i=1,2,\ldots,n$ labeling the rows, λ_i^k being the lengths of the corresponding rows). We define the quantity

$$U\begin{bmatrix}\lambda^3\\ \lambda^1\end{bmatrix}\begin{bmatrix}\lambda^4\\ \lambda^2\end{bmatrix} = \prod_{i,j=1}^{n}\prod_{k_i=\lambda_i^3+1}^{\lambda_i^4}\prod_{l_j=\lambda_j^1+1}^{\lambda_j^2}\left(1+\frac{1}{k_i-i-l_j+j}\right)$$

$$= \prod_{1=i\le j}^{n} \frac{(\lambda_i^4-\lambda_j^1-i+j)!(\lambda_i^3-\lambda_j^2-i+j)!}{(\lambda_i^4-\lambda_j^2-i+j)!(\lambda_i^3-\lambda_j^1-i+j)!} \times \prod_{1=i<j}^{n} \frac{(\lambda_i^2-\lambda_j^4-i+j-1)!(\lambda_i^1-\lambda_j^3-i+j-1)!}{(\lambda_i^1-\lambda_j^4-i+j-1)!(\lambda_i^2-\lambda_j^3-i+j-1)!}, \tag{A1}$$

Let, further, $[\lambda^2]$ and $[\lambda^3]$ be the Young schemes with $\lambda_i^2=\min(m_{in},m'_{in-1})$, $\lambda_i^3=\max(m_{in},m'_{in-1})$. In the notations of this paper, the result of Ref. 3 allows us to write

$$\begin{bmatrix}[m]_n & p & [m']_n\\ [m]_{n-1} & q & [m']_{n-1}\end{bmatrix} = \left(\frac{A[m]_n A[m']_{n-1}(p-q)!}{A[m']_n A[m]_{n-1}}\right)^{1/2} \left(U\begin{bmatrix}\lambda^3\\ m\end{bmatrix}_{n-1}\begin{bmatrix}m'\end{bmatrix}_n\begin{bmatrix}\lambda^2\end{bmatrix}\right)^{-1} \times \left(U\begin{bmatrix}\lambda^2\\ m\end{bmatrix}_{n-1}\begin{bmatrix}m'\end{bmatrix}_n\begin{bmatrix}\lambda^2\end{bmatrix} U\begin{bmatrix}\lambda^3\\ m\end{bmatrix}_{n-1}\begin{bmatrix}m'\end{bmatrix}_n\begin{bmatrix}\lambda^3\end{bmatrix}\right)^{1/2} F_{p_1}, \tag{A2}$$

where

$$A[\lambda] = (\textstyle\sum\lambda_i)!/f_{[\lambda]},$$

$f_{[\lambda]}$ being the dimension of the representation $[\lambda]$ of the symmetric group on $\sum_i\lambda_i$ symbols. F_{p_1} is the sum of the coefficients of those permutations in the expression

$$\prod_{k=p_1+1}^{p_1+p_2} \uparrow \prod_{l=1}^{p_1}\left(\mathscr{E}+\frac{(kl)}{x_k-y_l}\right), \tag{A3}$$

which the symbols $k\le p_1$ substitutes by the symbols $l>p_1$. $\mathscr{E}$ in (A3) is the unit element of the symmetric group and (lk), the transposition of symbols l and k. $\uparrow$ indicates that the order of multipliers with respect to label l is the same for each k. F_{p_1} is the symmetric function on two sets of variables x_k and y_l. For calculating the i.f. according to (A2), we must substitute the values of the function F_{p_1} with the x equal to

$k_i - i$ ($i = 1, 2, \ldots, n$; $k_i = \lambda_i^3 + 1, \lambda_i^3 + 2, \ldots, m'_{in}$; $p_2 = \sum_{i=1}^n (m'_{in} - \lambda_i^3)$) and the y to the $l_j - j$ ($j = 1, 2, \ldots, n-1$; $l_j = m_{jn-1} + 1, m_{jn-1} + 2, \ldots, \lambda_j^2$; $p_1 = \sum_{j=1}^{n-1} (\lambda_j^2 - m_{jn-1})$).

Extending the definition of F_{p_1}, we can define the set of bisymmetric functions F_s ($s = 0, 1, \ldots, p_1$), F_s being the sum of the coefficients of those permutations in (A3) in which s arbitrary symbols from the set $1, 2, \ldots, p_1$ are substituted by the symbols from another set $p_1 + 1, p_1 + 2, \ldots, p_1 + p_2$. It can be shown, that the following set of equations hold for the F_s:

$$\sum_s \frac{(p_1 - s)!(p_2 - s)!}{(p_1 - v_1)!(v_1 - s)!(p_2 - v_2)!(v_2 - s)!} F_s = V_{v_1 v_2}^{p_1 p_2}, \tag{A4}$$

where

$$V_{v_1 v_2}^{p_1 p_2} = \sum_{(1)} \sum_{(2)} \prod_{i=p_1+v_2+1}^{p_1+p_2} \prod_{k=p_1+1}^{p_1+v_2} \prod_{l=1}^{v_1} \prod_{j=v_1+1}^{p_1} \times \left(1 + \frac{1}{x_i - x_k}\right)\left(1 + \frac{1}{x_k - y_l}\right)\left(1 + \frac{1}{y_l - y_j}\right). \tag{A5}$$

v_1 and v_2 can take on arbitrary values from the intervals $0 \leq v_1 \leq p_1$ and $0 \leq v_2 \leq p_2$, respectively. The first summation in (A5) is taken with respect to permutations, one from each left coset of the group of permutations of indices $1, 2, \ldots, p_1$ with respect to the subgroup of permutations of indices $1, \ldots, v_1$ and $v_1 + 1, \ldots, p_1$, within the two sets. The second summation is analogous to the first one, the group being the permutation group of the symbols $p_1 + 1, \ldots, p_1 + p_2$, and the subgroup having as its elements the permutations within the two sets $p_1 + 1, \ldots, p_1 + v_2$ and $p_1 + v_2 + 1, \ldots, p_1 + p_2$.

Taking the different sets of $(p_1 + 1)$ equations from the extended set (A4), we obtain different expressions for F_{p_1}. Thus, if we take the equations with $v_2 = p_2$ and v_1 varying from 0 to p_1, we have

$$F_{p_1} = \sum_{v_1=0}^{p_1} (-1)^{p_1 - v_1} V_{v_1 p_2}^{p_1 p_2}. \tag{A6}$$

The value of the bisymmetric function $V_{v_1 p_2}^{p_1 p_2}$, the arguments taking the mentioned values, is equal to

$$\sum_{[r]} U\begin{bmatrix}[\lambda^3] & [m']_n \\ [r] & [\lambda^2]\end{bmatrix} U\begin{bmatrix}[r] & [\lambda^2] \\ [m]_{n-1} & [r]\end{bmatrix}, \quad \sum_i (\lambda_i^2 - r_i) = v_1. \tag{A7}$$

Using (A6)–(A7) for the F_{p_1}, Young's expression for the dimensions $f_{[\lambda]}$, and performing the simplifications needed, we obtain formula (9) for the i.f. under consideration.

On the other hand, solving equations (A4) with $v_1 = p_1$ and $v_2 = p_2 - p_1 - \alpha, p_2 - p_1 - \alpha + 1, \ldots, p_2 - \alpha$ using the values of x_i and y_k indicated above, one obtains

$$F_{p_2} = \sum_{[r]_n} \frac{(-1)^{y-\alpha} y!(p_2 - y)!}{(p - q)!(y - \alpha)!(p_1 + \alpha - y)!} \times U\begin{bmatrix}[\lambda^3] & [r]_n \\ [m]_{n-1} & [\lambda^2]\end{bmatrix} U\begin{bmatrix}[r]_n & [m']_n \\ [\lambda^3] & [r]_n\end{bmatrix}. \tag{A8}$$

Formulas (A2) and (A8) may be brought into the form equivalent to the result given by the Eq. (12) and (13).

APPENDIX B: RELATION BETWEEN RECOUPLING MATRICES AND ISOSCALAR FACTORS

According to the results of Ref. 5, the following relation holds between the elements of the recoupling matrix of four representations of U_n with three of them symmetric and the i.f.:

$$\begin{aligned}&\langle [m]_{n-1}\, q([m']_{n-1}), rp - q(r'); [m']_n \mid \\ &\quad \times \mid [m]_{n-1} r([m]_n), qp - q(p); [m']_n\rangle \\ &= \left(\frac{r!q!(p-q)!A[m]_{n-1}A[m']_n}{r'!p!A[m]_n A[m]_{n-1}}\right)^{1/2} \\ &\quad \times \begin{bmatrix}[m]_n & p & [m']_n \\ [m]_{n-1} & q & [m']_{n-1}\end{bmatrix}, \quad r = \sum_{i=1}^n m_{in} - \sum_{i=1}^{n-1} m_{in-1}, \\ &r' = p - q + r = \sum_{i=1}^n m'_{in} - \sum_{i=1}^{n-1} m'_{in-1},\end{aligned} \tag{B1}$$

$A[\lambda]$ is given in (A2).

The particular cases of this relation (when $p = q$ for U_n and for the semistretched i.f. of SU_3) have been obtained in Refs. 3 and 4. It can be seen that in the semistretched case of the i.f. ($m'_{nn} = m_{nn}$), the recoupling matrix goes over into the one of U_{n-1}. A particular case of this matrix (with $p = q$ and $m'_{nn} = m_{nn}$) gives the matrix changing the canonical chains of subgroups in U_n.[3, 14] Equations (12) and (15) for the i.f. on the right-hand side of (B1) are more convenient to use than Eq. (9), because in the first case there remain only $n - 2$ summation parameters, instead of $n - 1$ as is in the second case.

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