

SYMMETRIC POLYNOMIALS AND THE CENTER OF THE SYMMETRIC GROUP RING

A.-A. A. JUCYS

Institute of Physics and Mathematics of the Academy of Sciences of Lithuanian SSR, Vilnius,
Lithuanian SSR, USSR

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The homomorphism of a special kind between the ring of symmetric polynomials and the center of the symmetric group ring is established. In the homomorphic mapping of the first ring on the second one the proper values of the images are the values of the corresponding symmetric polynomials with the variables substituted by the set of integers found from the corresponding Young diagram.

1. Introduction

The symmetric group is one of the most familiar and important finite groups in pure mathematics as well as in physical applications of the group theory. In this note a relation of special type between the theory of the symmetric functions and the theory of the symmetric group characters is presented. The known relation between these two theories, the Frobenius formula ([5], p. 67), which is the most appropriate representative of it, is a consequence of the more comprehensive Schur–Weyl relation between the theories of representations of the symmetric and full linear group [9]. The relation formulated in this paper is an intrinsic property of the symmetric group.

2. Preliminaries

We assume the group ring and the ring of symmetric polynomials to be defined over the field of complex numbers, although the result is valid under more restrictive conditions. In particular, the field of complex numbers may be replaced by the ring of integers.

Considered as an algebra, the ring of symmetric polynomials in $n-1$ variables has an infinite basis

$$\sigma_1^{k_1} \sigma_2^{k_2} \dots \sigma_{n-1}^{k_{n-1}} \quad (k_i = 0, 1, \dots; i = 1, 2, \dots, n-1), \quad (1)$$

Now we prove that

$$Z_{n-p} = \sigma_p(y_1, y_2, \dots, y_{n-1}) \quad (p=0, 1, \dots, n-1). \quad (8)$$

Here in the analogy with (3) we assume $\sigma_0(y_1, y_2, \dots, y_{n-1})$ to be defined as the identity element ε of the group ring of S_n , the identity permutation itself constituting the class with n 1-cycles.

The proof of (8) can be carried out by induction. Performing the corresponding operations on elements y_k involved in the definition of the symmetric functions (2) we convince ourselves that (8) is valid for S_2, S_3, S_4 . Assuming this valid for S_{n-1} we prove validity for S_n using the expression

$$\sigma_p(y_1, y_2, \dots, y_{n-1}) = \sigma_p(y_1, y_2, \dots, y_{n-2}) + \sigma_{p-1}(y_1, y_2, \dots, y_{n-2}) \cdot y_{n-1} \quad (p=0, 1, \dots, n-1) \quad (9)$$

(note that $\sigma_{-1} = 0 = \sigma_{n-1}(y_1, y_2, \dots, y_{n-2})$). The first summand on the right-hand side of (9) considered as an element of the group ring of S_{n-1} is, by the induction hypothesis, equal to Z_{n-1-p} . The summand considered as an element of the group ring of S_n , is the sum of all the permutations from Z_{n-p} having the symbol n in the 1-cycle. σ_{p-1} from the second term in (9) is, by the induction hypothesis, the sum of all the permutations of S_{n-1} having $n-p$ cycles. When multiplied by y_{n-1} , the resulting permutations of S_n have the same number of cycles the symbol n not occurring in the 1-cycle and each such permutation occurring only once. These facts follow from the multiplication rule of the elements of S_n : if

$$(i_1 i_2 \dots i_r) \dots (k_1 k_2 \dots k_s)$$

is the full cycle expression (including 1-cycles) of the permutation of S_{n-1} , then

$$\begin{aligned} (i_1 i_2 \dots i_r) \dots (k_1 k_2 \dots k_s) \cdot y_{n-1} = & (i_1 n i_2 \dots i_r) \dots (k_1 k_2 \dots k_s) + (i_1 i_2 n \dots i_r) \dots (k_1 k_2 \dots k_s) + \dots \\ & \dots + (i_1 i_2 \dots i_r n) \dots (k_1 k_2 \dots k_s) + \dots + (i_1 i_2 \dots i_r) \dots (k_1 n k_2 \dots k_s) + \\ & + (i_1 i_2 \dots i_r) \dots (k_1 k_2 n \dots k_s) + \dots + (i_1 i_2 \dots i_r) \dots (k_1 k_2 \dots k_s n). \end{aligned} \quad (10)$$

It follows that the expression (9) is the sum of all the permutations of S_n having $n-p$ cycles. Thus (8) is proved.

From the "fundamental theorem" mentioned above, and from the result of Farahat and Higman and equation (8) it follows that each element of the center of the group ring of S_n may be expressed as a symmetric polynomial in the set of $n-1$ elements (7) (obviously, not uniquely, if the order of the group is finite). It follows also that each polynomial symmetric in the elements (7) is a uniquely defined element of the center of the group ring. Thus, a special kind of homomorphism is established between the two rings. The main interesting property of this homomorphism is connected with the proper values of the images.

4. Proper values of the homomorphic images

It is well known that an element of the group ring belongs to its center if and only if it is represented in all the irreducible representations $[\lambda]$ by multiples of the identity matrices. In particular, the multipliers (or *proper values*) for the basic elements (4), the so-called *class multipliers* [6], are

$$\omega_{\alpha}^{[\lambda]} = \frac{h_{\alpha} \chi_{\alpha}^{[\lambda]}}{f_{\lambda}}, \quad (11)$$

where $\chi_{\alpha}^{[\lambda]}$ is the irreducible character, f_{λ} the dimension of the representation, and h_{α} the number of elements in the class (α) .

If we take the proper functions of the set of operators (7) for the basis of the irreducible representation $[\lambda]$ of S_n , then the elements of the basis are uniquely characterized [3] by the standard Young tableaux [6] of the Young diagram $[\lambda]$, the diagram itself characterizing the irreducible representation. The proper value $\langle \rho | y_k | \rho \rangle$ of y_k for the function, characterized by the standard tableaux ρ , is equal to the difference between the column index i_{k+1} and the row index j_{k+1} of the $(k+1)$ -th node in ρ [3]

$$\langle \rho | y_k | \rho \rangle = i_{k+1} - j_{k+1}. \quad (12)$$

The standard tableaux of the same diagram differ one from another only in the enumeration of their nodes by the symbols $2, 3, \dots, n$ [6]. It follows that each symmetric polynomial in the set of elements (7) is represented in $[\lambda]$ by a multiple of the identity matrix, the multiplier equal to the numerical value of the polynomial when the variables assume the values of the integers $(i_{k+1} - j_{k+1})$ for $n-1$ nodes of the Young diagram $[\lambda]$.

The results of the preceding section thus imply that the symmetric polynomials for calculating the proper values of the elements of the center of the group ring of S_n may be obtained. In particular, the class multipliers (11) and, consequently, the irreducible characters of S_n can be calculated with the help of the polynomials found in such way.

5. Remarks and examples

1. The set of the n integers $(i_{k+1} - j_{k+1})$ for a given Young diagram $[\lambda]$ is called in [7] the *content* of this diagram. In our formulation we excluded the node in the upper left corner of the diagram. However, the latter may be included in a trivial way, because for this node $(i_1 - j_1) = 0$ in all Young diagrams.

We note that the integers from the content of $[\lambda]$ enter differently here than they do either in the fundamental theorem of Young ([6], p. 38) or in the expression for the Young operators given in [2]. Certainly, the expression for the diagonal matrix element of an element of the center of the group ring, found from formula (3.30) of [2] as a function in n variables, will not have the form of a symmetric polynomial. As noticed in [3], this results from the validity of the last expression for any skew-representation, the irreducible one being only a particular case of these latter.

2. The expounded homomorphic mapping may be reformulated for the ring of symmetric polynomials in m variables with arbitrary $m > n - 1$. In such case one must assume that the image of σ'_k s with $k > n - 1$ is the zero of the group ring. Then, expressing the symmetric polynomial as a polynomial in σ 's, one finds the image and its proper values so as if the σ'_k s for $k \leq n - 1$ are the functions in $n - 1$ variables.

3. Instead of taking the elementary functions for the generating symmetric functions, we must take other sets of functions, for instance, ([5], [7]), *complete homogeneous* or *power sum* symmetric functions. Their homomorphic images in the center of the group ring of S_n are sets of generators of the center, different from two such known sets ([1], [4]).

4. If C_α is expressed as a polynomial $P(s_1, s_2, \dots, s_{n-1})$ in power sum symmetric functions

$$s_k = \sum_{i=1}^{n-1} y_i^k \quad (k=1, 2, \dots, n-1) \quad (13)$$

and Frobenius' notation ¹

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

is used for the Young diagram $[\lambda]$ ([5], p. 60) then it follows from (11) that the character of the class (α) is

$$\chi_\alpha^{[\lambda]} = \frac{P(t_1, t_2, \dots, t_{n-1}) f_\lambda}{h_\alpha}, \quad (14)$$

where

$$t_k = \sum_{i=1}^r \left[\sum_{l=1}^{a_i} l^k + (-1)^k \sum_{l=1}^{b_i} l^k \right]. \quad (15)$$

Hence the Bernoulli numbers enter into the expressions for irreducible characters of the symmetric group.

5. Let us take an example.

The Young diagram [5421] in Frobenius' notation is given by

$$\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}.$$

The content of this diagram is (we place the integers of the content in the corresponding nodes of diagram)

0	1	2	3	4
-1	0	1	2	
-2	-1			
-3				

¹ In this notation a_i is equal to the number of nodes in the i th row on the right from the node on the leading diagonal and b_i denotes the number of nodes in the i th column below the same node.

It is easy to verify that for the class (α') of S_n , having one 3-cycle and $n-3$ 1-cycles

$$C_{\alpha'} = s_2 - \frac{n(n-1)}{2} \varepsilon = \sum_{i=1}^{n-1} y_i^2 - \frac{n(n-1)}{2} \varepsilon. \quad (16)$$

It follows that in this case the class multipliers are equal to

$$\begin{aligned} \omega_{\alpha'}^{[\lambda]} &= \sum_{i=1}^r \left[\sum_{l=1}^{a_i} l^2 + \sum_{l=1}^{b_i} l^2 \right] - \frac{n(n-1)}{2} = \\ &= \frac{1}{6} \sum_{i=1}^r [a_i(a_i+1)(2a_i+1) + b_i(b_i+1)(2b_i+1)] - \frac{n(n-1)}{2}. \end{aligned} \quad (17)$$

For the representation [5421] we have

$$\omega_{\alpha'}^{[5421]} = -16.$$

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