A LETTER TO THE EDITOR

A NOTE ON ONE CONJECTURE

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Running head:

On one conjecture

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Let S_n be the group of all permutations of some set with n (—) elements. Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n)$ be the partition of n with $\mathcal{L}_1 \geqslant \mathcal{L}_2 \geqslant \dots \geqslant \mathcal{L}_n$ $(\mathcal{L}_j \geqslant 0, \sum_{j=1}^{n} \mathcal{L}_j = n)$ and let $K_{\mathcal{L}} = S_{\mathcal{L}_1} \times S_{\mathcal{L}_2} \times \dots \times S_{\mathcal{L}_n}$ be the direct product of the subgroups of S_n, acting on disjoint subsets of the basic set.

Denote by $\langle \mathcal{L} \rangle$ the representation of S_n, induced by the trivial representation of $K_{\mathcal{L}}$. Let $\{\beta\}$ denote the irreducible representation, defined by the partition $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ of n [1 - 3]. Natural question arises: what is the intertwining number $\langle \langle \mathcal{L} \rangle | \{\beta\} \rangle$ of the two representations, which in terms of characters has the expression

$$\langle \langle \mathcal{L} \rangle | \{\beta\} \rangle = \frac{1}{n!} \sum_{s \in S_n} \chi_s^{\langle \mathcal{L} \rangle} \chi_s^{\langle \beta \rangle}$$

The answer to the question is obtained directly from the special case of the Littlewood - Richardson rule ([3], p. 92). It will be formulated here in the form of diofantine equalities and inequalities.

Let m_{ij} be natural numbers (including zero). We will assume them to be arranged so as to form the triangular array (in conformity with works of theoretical physicists [4], where the Gelfand - Ceitlin bases of irreducible representations of unitary groups are frequently meant as being enumerated by analogous arrays):

$$\begin{array}{cccc}
m_{n1} & m_{n2} & \cdots & m_{nn} \\
m_{(n-1)1} & m_{(n-1)2} & \cdots & m_{(n-1)(n-1)} \\
\vdots & & \ddots & \\
m_{11} & & &
\end{array}$$

The following conditions must be satisfied by the numbers m_{ij}:

$$m_{nj} = \beta_j \qquad (1)$$

$$\sum_{j=1}^{i} m_{ij} = \sum_{j=1}^{i} \alpha_j \qquad (2)$$

$$m_{(i+1)j} \geqslant m_{ij} \geqslant m_{(i+1)(j+1)} \qquad (3)$$

The answer to the question raised above is: $\langle\langle\alpha\rangle|\{\beta\}\rangle$ is equal to the number of distinct solutions of (1) - (3). Thus it follows

Proposition. $\langle\langle\alpha\rangle|\{\beta\}\rangle \neq 0$ if and only if $\sum_{j=1}^{i} \alpha_j \leqslant \leqslant \sum_{j=1}^{i} \beta_j$ for each i.

This statement was proposed in [5], page 533, as a conjecture based "on grounds of the study of the table in Murnaghan's book" ([6], p. 154). Besides pointing out that the validity of the conjecture follows directly from the known results, we want to note the following. The statement seams never to have been explicitly formulated and used in the classical theory of representations of the symmetric groups [1 - 5]. In these classical presentations the linear ordering of the partitions of n "according to the first of nonequal rows" is used, while partial ordering, proposed in [5] may occure to be even more useful as implied by the Proposition.

To be more specific, the two orderings are as follows. The classical linear ordering is: $\alpha < \beta$ means that the first of nonvanishing differences $\beta_1-\alpha_1, \beta_2-\alpha_2, \ldots$ is positive (sometimes the inverse ordering is used). Snapper proposed the ordering:

$$(\alpha \trianglelefteq \beta) \equiv \underset{i}{\forall}\left(\sum_{j=1}^{i}\alpha_j \leq \sum_{j=1}^{i}\beta_j\right).$$

The latter one is a partial ordering for $n \geqslant 6$ because, for instance, neither $(n-k,k) \trianglelefteq (n-k+1,k-2,1)$, nor $(n-k+1,k-2,1) \trianglelefteq (n-k,k)$. k-?

Clearly, $\alpha \trianglelefteq \beta$ implies $\alpha \leq \beta$. In classical presentations essential is the use of the analogue of the Proposition with $\alpha \trianglelefteq \beta$ replaced by $\alpha \leq \beta$. Concluding, we quote Gian-Carlo Rota ([7], p. 341): "in a great many cases a more effective technique is to work with the natural order of the set" (instead of fitting it into some linear order).

REFERENCES

1. D.E. Rutherford, "Substitutional Analysis", Hafner, New York, 1968.

2. B.L. Van der Waerden, "Modern Algebra", Vol. 2, Frederick Ungar Publishing Co., New York, 1949.

3. D.E. Littlewood, "The Theory of Group Characters", Clarendon Press, Oxford, 1958.

4. J.D. Louck, Recent progress toward a theory of tensor operators in the unitary groups, Amer. J. Phys. 38 (1970), 3-42.

5. E. Snapper, Group characters and nonnegative integral matrices, J. Algebra 19 (1971), 520-535.

6. F.D. Murnaghan, "The Theory of Group Representations", Dover, New York, 1963.

7. G.-C. Rota, On the foundations of combinatorial theory, I. Theory of Mobius functions, Z. Wahrscheinlichkeitstheorie 2 (1964), 340-368.