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Generating functions, Schensted algorithm, and quater indices of permutations

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The generating function for multiplicities of irreducible representations [λ] of the symmetrical group S_n in the polynomial basis with n variables, multiplied by the product of factors $(1-y^i)$ with i running from 1 to n, are considered. Using the Schensted algorithm and one of the MacMahon's results, it is shown that the coefficient of y^k in such a product is defined uniquely by the standard Young tableaux of the Young diagram [λ]. The corresponding rule has been found that each standard tableau adds a unit contribution to one of the coefficients.

1. The symmetric group S_n consists of a set of $n!$ different permutations of n objects. In the case of a many-particle quantum-mechanical problem, the objects rearranged are particle coordinates. Irreducible representations of the group S_n are specified by the Young diagrams [λ].

In Ref. [1], a generating function has been found for multiplicities of irreducible representations $[\lambda] \equiv [\lambda_1, \lambda_2, \ldots, \lambda_n]$ of the symmetric group S_n in the basis of quantum states of identical particles with their one-particle wave functions transformed by irreducible representations of the unitary group SU_2. As a factor in this generating function, the following expression occurs:

$$P_{[\lambda]}(y) \equiv y^{p(\lambda)} \prod_{d=1}^{n} \left(1 - y^{h(d,\lambda)}\right)^{-1} = \sum_{k=0}^{\infty} C_{[\lambda]}(k) y^k, \quad (1)$$

where $p(\lambda) = \sum_{i=1}^{n} (i-1)\lambda_i$ and $h(d,\lambda)$ is the hook length of the d-th cell (in arbitrary enumeration) of the Young diagram [λ], i. e., the sum of unity and the numbers in the cells to the right in the same row and to the bottom in the same column of the Young diagram [λ]. The expression (1) itself is the generating function for multiplicities $C_{[\lambda]}(k)$ of irreducible representations [λ] in the basis set of homogeneous polynomials of n variables with the given total power k (see Ref. [1]). In this communication, we consider the following polynomial (cf. Eq. (1)):

$$M_{[\lambda]}(y) \equiv y^{p(\lambda)} \prod_{d=1}^{n} \left(1 - y^{h(d,\lambda)}\right)^{-1} \prod_{i=1}^{n} (1 - y^i)$$

$$= \sum_{k=1}^{n(n-1)/2} D_{[\lambda]}(k) y^k. \quad (2)$$

It will be shown further that each standard Young tableau of the Young diagram [λ] makes a unit contribution to one of the coefficients $D_{[\lambda]}(k)$. Therefore, it is natural to expect the result to be useful in distinguishing between repetitive identical irreducible representations.

2. While deriving the result of the present communication, the MacMahon's result concerning the number of permutations with the given quater index ([2], Sec. 3) and the Schensted algorithm [2–4] will be used. We shall now describe them briefly. Let us introduce the designation

$$(y)_m = \prod_{i=1}^{m} (1 - y^i). \quad (3)$$

Let μ_i be positive integers and $\sum_{i=1}^{p} \mu_i = n$. Under the quater index of the permutation $\sigma \equiv \xi_1 \xi_2 \ldots \xi_n$ of a multiset $\{1^{\mu_1} 2^{\mu_2} \ldots p^{\mu_p}\}$ we understand the sum

$$\mathrm{ind}(\sigma) = \sum_{i=1}^{n} \chi(\xi_i), \quad (4)$$

where $\chi(\xi_i) = i$ if $\xi_i > \xi_{i+1}$, and $\chi(\xi_i) = 0$ otherwise. For instance, for the permutation of the multiset $\{1^2\, 2^3\, 3^2\, 4^1\}$

$$\sigma' \equiv 2\,3\,1\,2\,1\,2\,4\,3 \tag{5}$$

the quater index equals $\mathrm{ind}\,(\sigma') = 2+4+7 = 13$. The MacMahon's result is the statement that the number of permutations with the quater index equal to k is specified by the coefficient C_k in the expansion

$$\frac{(y)_n}{\prod_{i=1}^{p} (y)_{\mu_p}} = \sum_{k=1}^{n} C_k y^k. \tag{6}$$

The Schensted algorithm for each permutation σ of the multiset $\{1^{\mu_1}2^{\mu_2}\ldots p^{\mu_p}\}$ sets one-to-one correspondence with two standard Young tableaux P, Q of one and the same Young diagram. By this, in the table P number i appears μ_i times, and in the table Q each of the numbers $1, 2, \ldots, n$ appears once. For instance, for permutation σ' (5) this algorithm yields the following two tables*):

$$P = \begin{matrix} 1\,1\,2\,3 \\ 2\,2\,4 \\ 3 \end{matrix}\,, \quad Q = \begin{matrix} 1\,2\,6\,7 \\ 3\,4\,8 \\ 5 \end{matrix}\,. \tag{7}$$

The Schensted algorithm is recurrent in the sense that after constructing the tables P' and Q' for the first i numbers of permutation σ, the algorithm brings the following $(i+1)$-th number (let this number be q) according to a certain rule. Namely, number q is placed in the first row of P', into the first cell from the left occupied by the smallest number greater than q in this row. If there is no such number, then a new cell is added from the right to the first row of the tableaux P' and Q', and then number q is put in P' and number $(i+1)$ in Q'. The number pushed out of the first row, if any, is placed in the second by the same rule, and so on. For instance, for σ' (5) at $i = 7$,

$$P' = \begin{matrix} 1\,1\,2\,4 \\ 2\,2 \\ 3 \end{matrix}\,, \quad Q' = \begin{matrix} 1\,2\,6\,7 \\ 3\,4 \\ 5 \end{matrix}\,. \tag{8}$$

The last number 3 in σ' (5) is introduced by the above rule, so that from the pair (8) the pair of tableaux (7) is formed.

3. Let $x_Q(i) = i$, if number $(i+1)$ in the standard Young tableau (containing no repeatitive numbers) is beneath number i or to the left of it, and $x_Q(i) = 0$ otherwise. Let us state the main result of the present communication as the following Theorem.

Theorem. The polynomial $M_{[\lambda]}(y)$ (see Eq. (2)) is equal

$$\sum_{Q \in [\lambda]} y^{\sum_{i=1}^{n} x_Q(i)} = M_{[\lambda]}(y), \tag{9}$$

where the sum is over all standard Young tableaux Q of one and the same Young diagram $[\lambda]$.

As an illustration to Eq. (9), note that the standard tableau Q of Eq. (7) makes a contribution to the polynomial $M_{[4,3,1]}$ equal to y^{2+4+7}.

For proving this Theorem, we shall use an important property of the Schensted algorithm proved in Ref. [4] as Theorem 1, which states that for any permutation $\xi_1, \xi_2, \ldots, \xi_n$ of the multiset $\{1^{\mu_1}2^{\mu_2}\ldots p^{\mu_p}\}$, for all $i = 1, 2, \ldots, n$, the equalities

$$\chi(\xi_i) = x_Q(i). \tag{10}$$

are satisfied.

We shall demonstrate that the expressions in the left-hand side of Eq. (9) and the expressions in the right-hand side of Eq. (9) satisfy the same set of linear inhomogeneous equations. Since the matrix of that set will appear to be nondegenerate, from this fact the equality (9) will follow.

We shall confine ourselves to considering only such multisets $\{1^{\mu_1}2^{\mu_2}\ldots n^{\mu_n}\}$ where appearance multiplicities μ_i of positive integers i satisfy the inequalities

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_n. \tag{11}$$

From the MacMahon's result and the property (10) of the Schensted algorithm described above, the set of equations

$$\frac{(y)_n}{\prod_{i=1}^{n} (y)_{\mu_i}} = \sum_{[\lambda]} K_{\mu\lambda} \sum_{Q \in [\lambda]} y^{\sum_{i=1}^{n} x_Q(i)} \tag{12}$$

follows, where $K_{\mu\lambda}$ are matrix elements composing jointly the Costky matrix. Indeed, applying the Schensted algorithm to all permutations of the multiset $\{1^{\mu_1}2^{\mu_2}\ldots n^{\mu_n}\}$ we find that permutations with different P but with the same Q make, due to Eq. (10), identical contributions. The number of such permutations with different P at appearance multiplicities μ_i of the integers i equals the matrix element $K_{\mu\lambda}$ of the Costky matrix [5, 6].

On the other hand, denoting by $h(\mu_i)$ the Schur function of the variables $x_1, x_2, \ldots, x_n$ corresponding to the symmetric representation specified by the single-row Young diagram $[\mu_i]$ and denoting through $\{\lambda\}$ the Schur function specified by an arbitrary Young diagram $[\lambda]$, we have

$$\prod_i h(\mu_i) = \sum_{[\lambda]} K_{\mu\lambda}\{\lambda\}. \tag{13}$$

*) In the examples, the borders of Young diagram cells are omitted.

Substituting in Eq. (13) the powers of one variable $y^0, y^1, y^2, \ldots$ instead of the variables $x_1, x_2, x_3, \ldots$, we get (see Refs. [5, 6])

$$\frac{1}{\prod_{i=1}^{n} (y)_{\mu_i}} = \sum_{[\lambda]} K_{\mu\lambda} P_{[\lambda]}(y). \tag{14}$$

After multiplying both sides of Eq. (14) by $(y)_n$, we come to the following equality:

$$\frac{(y)_n}{\prod_{i=1}^{n} (y)_{\mu_i}} = \sum_{[\lambda]} K_{\mu\lambda} M_{[\lambda]}(y). \tag{15}$$

Since the Costky matrix $\|K_{\mu\lambda}\|$ is nondegenerate [5, 6], the equalities (9) follow from Eqs. (15) and (12). This completes the proof of Theorem.

4. In conclusion, we present an example illustrating the proved equality for the Young diagram [3, 2, 1]. According to the definition (2),

$$M_{[3,2,1]} = \frac{y^4(1-y)(1-y^2)(1-y^3)(1-y^4)(1-y^5)(1-y^6)}{(1-y^5)(1-y^3)^2(1-y)^3} = y^4 + 2y^5 + 2y^6 + 3y^7 + 3y^8 + 2y^9 + 2y^{10} + y^{11}. \tag{16}$$

All the 16 standard Young tableaux Q of the Young diagram [3, 2, 1] are given below. Under each table, a monomial from the sum in the right-hand side of Eq. (9) is written out. The sum $\sum_{i=1}^{n} x_Q(i)$ is indicated in the power of the variable y.

123	124	134	125	135	123	124	134
45	35	25	34	24	46	36	26
6	6	6	6	6	5	5	5
y^{3+5}	y^{2+4+5}	y^{1+4+5}	y^{2+5}	y^{1+3+5}	y^{3+4}	y^{2+4}	y^{1+4}

125	135	145	126	136	126	136	146
36	26	26	34	24	35	25	25
4	4	3	5	5	4	5	3
y^{2+3+5}	y^{1+3+5}	y^{1+2+5}	y^{2+4}	y^{1+3+4}	y^{2+3}	y^{1+3}	y^{1+2+4}

Indeed, as it is stated by Theorem, the sum of all the monomials indicated here equals the expression (16).

References

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