INNER PRODUCT OF SCHUR FUNCTIONS

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1. In [1, 2] Littlewood introduces and studies the inner product of Schur functions, defined as follows. Let $\chi_{(\rho)}^{[\lambda]}$ denote the character of conjugacy class (ρ) in the irreducible representation $[\lambda]$ of the symmetric group S_n and let

$$\chi_{(\rho)}^{[\lambda]}\chi_{(\rho)}^{[\mu]}=\sum_{\nu} g_{\lambda\mu\nu}\chi_{(\rho)}^{[\nu]}. \tag{1}$$

Then the inner product, denoted by the symbol $\circ$, is defined as follows:

$$\{\lambda\}\circ\{\mu\}=\sum_{\nu} g_{\lambda\mu\nu}\{\nu\}. \tag{2}$$

Thus, the study of inner products of Schur functions is equivalent to the study of Kronecker products of representations of symmetric groups. However, the Schur functions introduced here seem to be somewhat artificial. In the definition (2) all the Schur functions $\{\lambda\}$, $\{\mu\}$, $\{\nu\}$ depend on the same variables. In the following point of the present note it is shown that the problem can be formulated naturally in terms of the theory of Schur functions if it is interpreted as the problem of decomposing Schur functions of the Cartesian product of two sets of variables in terms of the products of the Schur functions of each of the two sets of variables separately. A new proof of the result of Robinson–Taulbee on the coefficients $g_{\lambda\mu\nu}$, based on this interpretation is given in point 3. In the last point the $g_{\lambda\mu\nu}$ are expressed in terms of the number of nonnegative integral cubical matrices with given sums of matrix elements in planes.

2. By lower case Greek letters we denote partitions $\alpha\equiv(\alpha_1\geqslant\alpha_2\geqslant\ldots\geqslant\alpha_n)$ of the number $n=\sum\alpha_i$, and also the Ferrera–Young diagrams corresponding to them. By the same letter in ordinary, curly, and square brackets we denote respectively the conjugacy classes of elements of the group S_n, the Schur function, and the irreducible representation of the group S_n, defined by this partition. In what follows we shall indicate sets of variables in the notation for symmetric functions. Thus, if $x=\{x_1, x_2, \ldots, x_a\}$, then (cf. [3, p. 86])

$$\{\lambda\}_x\equiv\{\lambda\}(x_1, x_2, \ldots, x_a)=\frac{1}{n!}\sum_{\alpha} k_{(\alpha)}\chi_{(\alpha)}^{[\lambda]}S_{(\alpha),x}, \tag{3}$$

where

$$S_{(\alpha),x}\equiv S_{(\alpha)}(x_1, x_2, \ldots, x_a)=\prod_{i=1}^{n}(x_1^{\alpha_i}+x_2^{\alpha_i}+\ldots+x_a^{\alpha_i}).$$

In (3) $k_{(\alpha)}$ is equal to the number of elements in the conjugacy class (α) of the group S_n. The decomposition inverse to (3) is [3, p. 86]

$$S_{(\alpha),x}=\sum_{\lambda}\chi_{(\alpha)}^{[\lambda]}\{\lambda\}_x. \tag{4}$$

The set $xy=\{x_1y_1, \ldots, x_1y_b, x_2y_1, \ldots, \ldots, x_ay_b\}$ is the Cartesian product of two sets of variables $x=\{x_1, x_2, \ldots, x_a\}$, $y=\{y_1, y_2, \ldots, y_b\}$. Obviously

$$S_{(\alpha),xy}=S_{(\alpha),x}S_{(\alpha),y}. \tag{5}$$

It follows from (3), (4), and (5) that

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$$\{\nu\}_{xy}=\frac{1}{n!}\sum_{\alpha,\lambda,\mu}k_{(\alpha)}\chi^{[\lambda]}_{(\alpha)}\chi^{[\mu]}_{(\alpha)}\chi^{[\nu]}_{(\alpha)}\{\lambda\}_x\{\mu\}_y. \tag{6}$$

Multiplying both sides of (1) by $k_{(\alpha)}\chi^{[\nu]}_{(\alpha)}$, summing over α, and using the orthogonality property of characters of groups, we find

$$\frac{1}{n!}\sum_{\alpha}k_{(\alpha)}\chi^{[\lambda]}_{(\alpha)}\chi^{[\mu]}_{(\alpha)}\chi^{[\nu]}_{(\alpha)}=g_{\lambda\mu\nu}. \tag{7}$$

Thus, from (6) and (7) we have

$$\{\nu\}_{xy}=\sum_{\lambda,\mu}g_{\lambda\mu\nu}\{\lambda\}_x\{\mu\}_y. \tag{8}$$

We note that (8) can be interpreted as giving the decomposition of an irreducible character of the general linear group GL(a, b) upon restricting the corresponding representations to the subgroup GL(a) × GL(b).

3. Using (8), we get an algorithm equivalent to the method of Robinson and Taulbee [1, 4], for finding the numbers $g_{\lambda\mu\nu}$. We shall consider the pairs of indices of products of variables x_iy_j, x_sy_r from xy to be ordered by the following conventions:

$$\langle i, j\rangle<\langle s, r\rangle, \quad \text{if} \quad i<s, \tag{9a}$$

$$\langle i, j\rangle\leqslant\langle i, r\rangle, \quad \text{if} \quad j\leqslant r. \tag{9b}$$

It is well known [3, 5] that the coefficient of the monomial $x_1^{\varkappa_1}x_2^{\varkappa_2}\ldots x_a^{\varkappa_a}$ in the twisted Schur function defined by the twisting diagram $\alpha-\beta$ (and in the ordinary Schur function, as the special case $\beta=0$), is equal to the number of twisted tableaux of the diagram $\alpha-\beta$ and of weight $\varkappa$, which are obtained by filling the cells of this twisted diagram $\varkappa_i$ with numbers i (i = 1, 2, ..., a) in nondecreasing order from left to right and strictly increasing from top to bottom in each column. With the goal of finding the coefficients in the expression $\{\nu\}_{xy}$ we shall fill the cells of the diagram ν with pairs of indices, ordered according to (9).

The coefficient $\langle\eta_1, \eta_2, \ldots, \eta_a\rangle_{\nu\mu}$ of $x_1^{\eta_1}x_2^{\eta_2}\ldots x_a^{\eta_a}\{\mu\}_y$ in $\{\nu\}_{xy}$ is found as follows. We consider those generalized tableaux the diagram ν, in which the first indices of pairs <i, j> are identically situated, with the first η_i indices equal to i for each i = 1, 2, ..., a. Let $\nu^{(i)}-\nu^{(i-1)}$ be the twisted subdiagram of the diagram ν, whose cells are occupied by the η_i pairs of indices with first indices equal to i. The requirements for filling the cells of the twisted diagram $\nu^{(i)}-\nu^{(i-1)}$ should be satisfied by the location of the second indices of the pairs (the indices of the y's) in accord with their ordering (9b). It follows from this that these tableaux make a contribution to the coefficient sought, equal to the coefficient of $\{\mu\}_y$ in the product of twisted Schur functions $\prod_{i=1}^{n}\{\nu^{(i)}-\nu^{(i-1)}\}_y$. Summing the contributions of all possible arrangements of first indices (respectively summing over all possible $\nu^{(i)}$, i = 1, 2, ..., a), and expressing the symmetric functions which appear in terms of the Schur functions $\{\mu\}_y$, we find $\langle\eta_1, \eta_2, \ldots, \eta_a\rangle_{\nu\mu}$.

One of the methods of expressing sums of products $\prod\{\nu^{(i)}-\nu^{(i-1)}\}_y$ in terms of Schur functions, leading to the method of Robinson and Taulbee, is the following. According to the Littlewood–Richardson rule [5, Chap. 1, Sec. 9] we decompose $\{\nu^{(i)}-\nu^{(i-1)}\}_y$ (i = 1, 2, ..., a) into a linear combination of Schur functions (ordinary, not twisted ones) and we find their products again using the same Littlewood–Richardson rule, finding, after summation, the coefficient $\langle\eta_1, \eta_2, \ldots, \eta_a\rangle_{\nu\mu}$.

As the last step we have

$$g_{\lambda\mu\nu}=\sum_{\sigma\in S_a}(-1)^{\sigma}\langle\lambda_1+a-\sigma(a), \lambda_2+a-1-\sigma(a-1), \ldots, \lambda_a+1-\sigma(1)\rangle_{\nu\mu}, \tag{10}$$

where $(-1)^{\sigma}$ = +1, −1, respectively, for even and odd permutations. Here we have used the familiar procedure for decomposing a monogenic symmetric function in terms of Schur functions [3, 5].

4. Now we proceed to finding a more symmetric expression for $g_{\lambda\mu\nu}$. We introduce a third set of variables $z = \{z_1, z_2, \ldots, z_c\}$ and let xyz be their Cartesian product. We have

$$\frac{1}{\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}(1-x_i y_j z_k)} = \sum_{\alpha,\beta,\gamma} p_{\alpha\beta\gamma} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_a^{\alpha_a} y_1^{\beta_1} y_2^{\beta_2} \ldots y_b^{\beta_b} z_1^{\gamma_1} z_2^{\gamma_2} \ldots z_c^{\gamma_c} = \sum_{n=0}^{\infty} h_{n,\,xyz}. \tag{11}$$

Here

$$p_{\alpha\beta\gamma} \equiv p_{\alpha_1\alpha_2 \ldots \alpha_a \beta_1 \beta_2 \ldots \beta_b \gamma_1 \gamma_2 \ldots \gamma_c}$$

is equal to the number of nonnegative integral cubical matrices $\|M_{ijk}\|$ with the following sums of numbers in planes:

$$\alpha_i = \sum_{j=1}^{b}\sum_{k=1}^{c} M_{ijk},\quad \beta_j = \sum_{i=1}^{a}\sum_{k=1}^{c} M_{ijk},\quad \gamma_k = \sum_{i=1}^{a}\sum_{j=1}^{b} M_{ijk}. \tag{12}$$

The homogeneous symmetric function $h_{n,xyz}$ is the Schur function defined by the partition consisting of one part (equal to n). The corresponding representation [n] of the group S_n is trivial with $\chi^{[n]}_{(\alpha)} = 1$ for all (α). It follows from (7) and the orthogonality properties of characters that

$$g_{[n]\,\lambda\mu} = \delta_{\lambda\mu}. \tag{13}$$

From (8) and (13) we get

$$h_{n,\,xyz} = \sum_{\nu} \{\nu\}_{xy}\{\nu\}_z = \sum_{\lambda,\,\mu,\,\nu} g_{\lambda\mu\nu}\{\lambda\}_x\{\mu\}_y\{\nu\}_z. \tag{14}$$

Again using the familiar decomposition of monogenic symmetric functions in terms of Schur functions, from (11) and (14) we find

$$g_{\lambda\mu\nu} = \sum_{\sigma\in S_a,\ \tau\in S_b,\ \varphi\in S_c} (-1)^{\sigma+\tau+\varphi} p_{\sigma(\lambda)\,\tau(\mu)\,\varphi(\nu)}. \tag{15}$$

The following abbreviations are used here: the parts of the partition $\sigma(\lambda)$ are $\lambda_1+a-\sigma(a)$, $\lambda_2+a-1-\sigma(a-1), \ldots, \lambda_a+1-\sigma(1)$, and analogously for $\tau(\mu)$ and $\varphi(\nu)$. It is easy to find the numbers $P_{\alpha\beta\gamma}$ from the expressions for $h_{n,\,xyz}$ in terms of products of polynomial symmetric functions

$$h_{n,\,xyz} = \frac{1}{n!}\sum_{\rho} k_{(\rho)} S_{(\rho),\,x} S_{(\rho),\,y} S_{(\rho),\,z}. \tag{16}$$

Suppose in the partition ρ there are $r_i(\rho)$ parts equal to i. Then

$$k_{(\rho)}/n! = \Big(\prod_{i=1}^{n} i^{r_i(\rho)} r_i(\rho)!\Big)^{-1}$$

and from (16) we get

$$p_{\alpha\beta\gamma} = \sum \prod_{i=1}^{n} \frac{(r_i(\rho)!)^2}{i^{r_i(\rho)} \prod_{j=1}^{n} a_{ij}! \prod_{s=1}^{n} b_{is}! \prod_{r=1}^{n} c_{ir}!}. \tag{17}$$

Here one has summed over the parameters satisfying the following equations:

$$\sum_{j=1}^{a} a_{ij} = \sum_{s=1}^{b} b_{is} = \sum_{r=1}^{c} c_{ir} = r_i(\rho),\quad \sum_{i=1}^{n} i r_i(\rho) = n,$$

$$\sum_{i=1}^{n} i a_{ij} = \alpha_j,\quad \sum_{i=1}^{n} i b_{is} = \beta_s,\quad \sum_{i=1}^{n} i c_{ir} = \gamma_r.$$

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