LETTER TO THE EDITOR

PARTITIONS INTO LIMITED PARTS: MULTIPLICITIES OF TERMS OF SHELLS OF EQUIVALENT PARTICLES

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In the case of enumerative generator functions that have the form of infinite sums $\sum_{l=0}^{\infty} c_l x^l$, the coefficients c_l are determined using the classical integral Cauchy theorem. It was in this way that Hardy and Ramanujan obtained the well-known formula for the quantity p(k), equal to the number of partitions of natural number k [1, 2]. In this case the generator function was written in the form

$$\sum_{k=0}^{\infty} p(k)x^k = \prod_{k=1}^{\infty}(1-x^k)^{-1}. \qquad (1)$$

However, a number of extremely important enumerative problems, including the above one, can be formulated in the language of generator functions that have the form of finite series. In particular, assume that $P_{m,n}(k)$ is the number of partitions of natural number k into not more than n parts, each of which does not exceed m. Obviously, $P_{m,n}(k) = p(k)$ for $k \leq m$, $k \leq n$. At the same time (see [1], Chapter 3),

$$\sum_{k=0}^{mn} P_{m,n}(k)x^k = \prod_{s=1}^{n} \frac{(1-x^{m+s})}{(1-x^s)}, \qquad (2)$$

i.e., the generator function for numbers $P_{m,n}(k)$, and hence for p(k), is expressed here in the form of a finite series. In the case of an arbitrary generator function

$$f(x) = \sum_{l=0}^{p} c_l x^l, \qquad (3)$$

having the form of a finite series, we assume the following simple elementary formula for c_l:

$$c_l = \frac{1}{p+1}\sum_{q=1}^{p+1} f(e^{2\pi iq/(p+1)})\, e^{-2\pi iql/(p+1)}. \qquad (4)$$

The validity of (4) can be established by substituting into it the values of $f[\exp(2\pi iq/(p+1))]$, expressed in the form of finite sums (3), and by using the familiar relation

$$\sum_{q=1}^{p+1} e^{2\pi iq(l'-l)/(p+1)} = (p+1)\delta_{l',l}. \qquad (5)$$

Employing Eq. (4) and expression (2), we obtain the following formula for the numbers $P_{m,n}(k)$:

$$P_{m,n}(k)=\frac{1}{mn+1}\sum_{q=1}^{mn+1}e^{-2\pi ikq/(mn+1)}\lim_{\varphi\to\pi q/(mn+1)}\prod_{s=1}^{n}\frac{(1-e^{i\varphi(m+s)})}{(1-e^{i\varphi s})}, \quad (6)$$

from which we find, after some elementary manipulations,

$$P_{m,n}(k)=\frac{1}{mn+1}\sum_{q=1}^{mn+1}\cos\frac{\pi q(mn-2k)}{mn+1}\lim_{\varphi\to\pi q/(mn+1)}\prod_{s=1}^{n}\frac{\sin[\varphi(m+s)]}{\sin[\varphi s]}. \quad (7)$$

Some of the formulas to follow will be given without proof (the author hopes to publish the proofs in the near future). Assume that $\bar{P}_{m,n}(k)$ is the number of partitions of natural number k into not more than n different parts, each of which does not exceed m ($m \geq n - 1$). We have

$$\sum_{l=0}^{mn}\bar{P}_{m,n}\left(l+\frac{n(n-1)}{2}\right)x^l=\prod_{s=1}^{n}\frac{(1-x^{m-n+1+s})}{(1-x^s)}. \quad (8)$$

From (8) we obtain, using Eq. (4),

$$\bar{P}_{m,n}\left(k+\frac{n(n-1)}{2}\right)=$$
$$=\frac{1}{mn+1}\sum_{q=1}^{mn+1}\cos\frac{\pi q(mn-n^2+n-2k)}{mn+1}\lim_{\varphi\to\pi q/(mn+1)}\prod_{s=1}^{n}\frac{\sin[\varphi(m-n+1+s)]}{\sin[\varphi s]}. \quad (9)$$

Expressions (6) and (9) enable us to find explicit formulas for the multiplicities $d_{j,n}(J)$ and $\bar{d}_{j,n}(J)$ of terms with a given overall moment of momentum J of n bosons and n fermions, respectively ($nj - J - 1 \geq 0$) in a cell with one-particle moment of momentum j:

$$d_{j,n}(J)=P_{2j,n}(nj-J)-P_{2j,n}(nj-J-1)=$$
$$=\frac{2}{2nj+1}\sum_{q=1}^{2nj+1}\sin\frac{\pi q(2J+1)}{2nj+1}\sin\frac{\pi q}{2nj+1}\times \quad (10)$$
$$\times\lim_{\varphi\to\pi q/(2nj+1)}\prod_{s=1}^{n}\frac{\sin[\varphi(2j+s)]}{\sin[\varphi s]},$$

$$\bar{d}_{j,n}(J)=\bar{P}_{2j,n}(nj-J)-\bar{P}_{2j,n}(nj-J-1)=$$
$$=\frac{2}{2nj+1}\sum_{q=1}^{2nj+1}\sin\frac{\pi q(2J+1)}{2nj+1}\sin\frac{\pi q}{2nj+1}\times \quad (11)$$
$$\times\lim_{\varphi\to\pi q/(2nj+1)}\prod_{s=1}^{n}\frac{\sin[\varphi(2j-n+s+1)]}{\sin[\varphi s]}.$$

We should note that papers [3-5] contain attempts to find expressions for the number of terms of n equivalent fermions. Paper [4] also investigated the asymptotic behavior of this quantity.

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