SUBCLASS ALGEBRA OF A FINITE GROUP

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Enumerated formulas are found for the number of subclasses in a particular class of conjugate elements of a finite group for a number of all subclasses and for a number of ambivalent subclasses. Simple matrix subalgebras of the subclass algebra are indicated in group algebra and the relations of orthogonality for irreducible representations of the subclass algebra are acquired. The interrelations between the subclass algebra and algebras of double coset classes are noted. Simple proofs of some of the known propositions are cited.

1. INTRODUCTION

The concept of a subclass was introduced by Wigner in works [1,2]. Assume that H is the subgroup of the G group. A relation of equivalency s ~ g is assigned to G if there is such an element $h \in H$ so that $s = hgh^{-1}$. G is subdivided with respect to this relation into mutually nonintersecting classes of equivalence which are also the subclasses of the G group with respect to the H subgroup, whose multitude is designated as P^{GH}. Hereafter, only the finite groups and their group algebras CG over the field of complex numbers C will be examined. Each subclass $p \in P^{GH}$ is placed in mutually unambiguous correspondence to the elements

$$p_C = \sum_{s \in p} s \tag{1}$$

of the CG group algebra, called the sum of the subclass. The product of the sums of the subclasses is expressed in the form of a linear combination of sums of subclasses (with whole number coefficients), so that the multitude of elements such as (1) is the basis of the subalgebra of the CG algebra, called the CP^{GH} subclass algebra.

Wigner in [1] (see also [2]) demonstrated that the frequencies f_{ν}^{λ} of appearance of irreducible representations ν of the H subgroup with limitation on its irreducible representations λ of the G finite subgroup do not exceed one when and only when the algebra of the CP^{GH} subclasses is commutative. This result is independently found by the authors of works [3,4]. The following relation is found in [5]:

$$\sum_{\lambda,\nu} (f_{\nu}^{\lambda})^2 = d^{GH}, \tag{2}$$

where $d^{GH} = |P^{GH}|$, i.e., d^{GH} is the number of subclasses. The equality (2) is one of two basic results of Wigner's subsequent work [6] (the equality II). There is mathematical literature dedicated to irreducible characters of subclass algebra [7,8]. The quite specific application of subclass algebra for enumerating multiparticle Feinman diagrams is demonstrated in works [9,10].

Simple subalgebras of the subclass algebra are indicated and examined in the next section of this work in CG; simple proofs of certain known propositions are constructed on this basis, and relations are found of the orthogonality for irreducible representations of the subclass algebra. Section 3 finds formulas for the number of sublcasses which contain in a particular class adjoined elements ρ of the G group and for partial

sums $\sum_{\lambda}(f_\lambda)^2$. Summing the first expressions in terms of ρ or the second in terms of λ, the following result is found:

$$d^{GH}=\frac{|G|}{|H|}\sum_{\rho}\frac{h_\rho}{g_\rho}, \tag{3}$$

where g_ρ is the number of elements in a class of adjoined elements ρ of the G group and h_ρ is the number of elements of the subgroup in this class (in the general case they may belong to different classes of the subgroup). Equality (3) for the special case when H is the centralizer of a specific element in the group was already deduced in [9,10]. Assume that k_ρ is the number of elements of the $K\subset G$ subgroup in the ρ class of the group. It is useful to compare (3) with the formula for the number of double coset classes of the group based on the H and K subgroups (section 3)

$$d^{G}_{KH}=\frac{|G|}{|H||K|}\sum_{\rho}\frac{h_\rho k_\rho}{g_\rho}. \tag{4}$$

In this same section it is shown that the number of subclasses in a class of adjoined elements α of the group is equal to $d^{G}_{N_\alpha H}$, where N_α is the centralizer of one of the elements of the α class.

In the last, fourth, section of the work a formula is deduced for the number of ambivalent subclasses d_a^{GH}

$$d_a^{GH}=\frac{1}{|H|}\sum_{h\in H}r_2(h), \tag{5}$$

where $r_2(h)$ is the number of solutions in the group of the equation $s^2=h^2$ $(s\in G)$.

As is evident from (3)-(5), not even the tables of characters of the group and subgroup are required to find d^{GH}, d^{G}_{KH} and d_a^{GH}; much less information is required (the tables of characters are required, in particular, when formula (2) is used for calculating d^{GH}).

2. BASIS OF A RELATIVE, REGULAR REPRESENTATION OF THE SUBCLASS ALGEBRA

Assume that λ assigns an irreducible representation of the dimensionality f_λ. In the CG group algebra the representation of λ is placed in a mutually unambiguous correspondence to the simple matrix subalgebra, whose basis is the $(f_\lambda)^2$ elements [11-13]

$$e^{\lambda}_{a,b}=\frac{f_\lambda}{|G|}\sum_{s\in G}\langle b|s^{-1}|a\rangle_\lambda s; \tag{6}$$

whose rules of multiplication are the following:

$$e^{\lambda}_{a,b}e^{\lambda'}_{c,d}=\delta_{\lambda\lambda'}\delta_{bc}e^{\lambda}_{a,d}. \tag{7}$$

Here $\langle a|t|b\rangle_\lambda$ designate the matrix element of the matrix which represents $t\in G$ in an irreducible representation of λ; a and b are the basis indexes. The following expansion for $s\in G$ [11-13] is the case:

$$s=\sum_{\lambda,a,b}\langle a|s|b\rangle_\lambda e^{\lambda}_{a,b}, \tag{8}$$

where the summing is performed in terms of all nonequivalent irreducible representations of the group. In the same way

$$|G|=\sum_{\lambda}(f_\lambda)^2 \tag{9}$$

elements (6) are the basis of the regular representation reduced to an irreducible form, whose other basis is comprised of the elements of the group themselves. The equalities

(6) and (8) assign the expansions of one basis through the other.

The presentation in [5] (pp. 15-17) (p. 18 in [5] should read "sufficient" instead of "necessary and sufficient.") will be basically followed in the rest of this section. The representations of the G group will be assumed to be relative with respect to the H subgroup, so that the basis indexes of the representations of the group below are assigned by three indexes ν, k, and i. To avoid multilevel indexes in the k index of these three, which differentiates and numbers the repeating irreducible representations of ν of the subgroup in the irreducible representation of the λ group, λ and ν will not be explicitly indicated, but from the context (including the formulas), it is hoped that it will be clear to which λ and ν this index is associated. This also applies to the i index, which numbers the basis of the irreducible representation ν of the subgroup. Primes will be used to differentiate the four indexes λ, ν, k, and i. Now (6) is written as

$$e^{\lambda}_{\nu ki,\nu' k' i'} = \frac{f_\lambda}{|G|} \sum_{s\in G} \langle \nu' k' i' | s^{-1} | \nu k i \rangle_\lambda s. \tag{10}$$

The following sums are introduced into examination:

$$O^{\lambda\nu}_{k,k'} = \sum_i e^{\lambda}_{\nu ki,\nu k'} = \frac{f_\lambda}{|G|} \sum_{s\in G} \left(\sum_i \langle \nu k' i | s^{-1} | \nu k i \rangle_\lambda \right) s. \tag{11}$$

Here at fixed λ and ν each of the indexes k and k' assume f^λ_ν different values; it follows from (7) that

$$O^{\lambda\nu}_{k,k'} O^{\lambda'\nu'}_{k'',k'} = \delta_{\lambda\lambda'} \delta_{\nu\nu'} \delta_{k'k''} O^{\lambda\nu}_{k,k'}. \tag{12}$$

The following is demonstrated [5].

Proposition 1. The totality of elements $O^{\lambda\nu}_{k,k'}$ determined by the equality (11) is the basis of the CP^{GH} subclass algebra.

Assume that $s\epsilon p\epsilon P^{GH}$ and assume that N_s is the centralizer [14] of the $s\epsilon G$ element in the H subgroup, i.e., N_s is the group which consists of elements of the H subgroup which commutate with s

$$N_s = \{ h\in H | hsh^{-1} = s \}. \tag{13}$$

When $s' = hsh^{-1}, h\epsilon H$, then $N_{s'} = hN_s h^{-1}$ and, therefore,

$$|N_s| = |N_{s'}| \equiv n_p, \tag{14}$$

i.e., the magnitude of the centralizers of all elements of this particular subclass p are identical. After breaking H into left coset classes in terms of the subgroup N_s, the following is found from (1), (13), (14):

$$p_C = \sum_{h\in H_\pi} hsh^{-1} = \frac{1}{n_p} \sum_{h\in H} hsh^{-1}, \tag{15}$$

where H_π is the system of representations of the left coset classes. It follows from (15) that the sum of the subclass commutates with each element of the subgroup

$$\forall h\in H, \quad p_C = hp_C h^{-1}. \tag{16}$$

Assume that E is any element from CG which commutates with each element of the H subgroup. E is unambiguously expanded in terms of the basis (10):

$$E = \sum_{\lambda,\nu,k,i,\nu',k',i'} \langle \nu k i | E | \nu' k' i' \rangle_\lambda e^{\lambda}_{\nu ki,\nu' k' i'}. \tag{17}$$

On the other hand, the following is the case for $h\epsilon H$:

$$h = \sum_{\lambda,\nu,k,i,\nu',k',i'} \langle \nu k i | h | \nu' k' i' \rangle_\lambda e^{\lambda}_{\nu ki,\nu' k' i'} = \tag{18}$$

$$= \sum_{\lambda,\nu,k,i,i'} \langle i|h|i'\rangle_\nu e^{\lambda}_{\nu ki,\nu ki'}. \tag{18}$$

As a result of (7) the coefficients of the expansion in terms of the basis (10) of the products of the elements of the CG group algebra are found from the rules of matrix multiplication. Since Eh = hE for each $h \in H$, then the matrices of expansions (17) and (18) are commutated for each $h \in H$ and, therefore, as a result of the Schur lemma [11,12,14], the following is the case:

$$\langle \nu ki|E|\nu' k' i'\rangle_\lambda = \delta_{\nu\nu'}\delta_{ii'} x^{\lambda\nu}_{k,k'}, \quad x^{\lambda\nu}_{k,k'} \in C. \tag{19}$$

Comparing (10), (17), and (19), it is found that

$$\forall h \in H, (hE = Eh) \Rightarrow E = \sum_{\lambda,\nu,k,k'} x^{\lambda\nu}_{k,k'} O^{\lambda\nu}_{k,k'}. \tag{20}$$

In the same way each element of the algebra of the CP^{GH} subclasses as a result of (16), (20) may be expanded in terms of the basis (11). It remains to show that there is a reverse expansion. When $h \in H$, then the following is the case:

$$\begin{aligned}\sum_i \langle \nu k' i|hs^{-1}h^{-1}|\nu ki\rangle_\lambda &= \sum_{i,i',i''} \langle \nu k' i|h|\nu k' i'\rangle_\lambda \langle \nu k' i'|s^{-1}|\nu ki''\rangle_\lambda \times \\ &\times \langle \nu ki''|h^{-1}|\nu ki\rangle_\lambda = \sum_{i',i''} \delta_{i'i''} \langle \nu k' i'|s^{-1}|\nu ki''\rangle_\lambda = \\ &= \sum_i \langle \nu k' i|s^{-1}|\nu ki\rangle_\lambda.\end{aligned} \tag{21}$$

After comparing (21) and (11), it is seen that the coefficients with elements of the same subclass in $O^{\lambda\nu}_{k,k'}$ are identical and in the same way $O^{\lambda\nu}_{k,k'}$ is the linear combination of the sums of subclasses. This completes the proof of proposition 1.

It follows from proposition 1 that CP^{GH} is isomorphous to the direct sum of simple matrix subalgebras, each of which is assigned by the λ, ν pair with $f^\lambda_\nu \neq 0$. In the same way the algebra of the subclasses is semisimplistic. A direct result is the equality (2) (when H is a trivial subgroup which consists of a single element - the single groups ε, then (2) is reduced to (9)).

The matrices of the coefficients of expansions of the sums of the subclasses in term of the basis (11)

$$p_C = \sum_{\lambda,\nu,k,k'} \langle k|p_C|k'\rangle^\lambda_\nu O^{\lambda\nu}_{k,k'} \tag{22}$$

are quasi-diagonal. Each unit on a diagonal corresponds to a fixed pair λ, ν and as a result of the multiplication rules (12) assigns the representation of the subclass algebr This representation is irreducible or the reducability would indicate the existence in CP^{GH} of a basis of a dimensionality less than d^{GH}. Assume $s \in p \in P^{GH}$; it follows from (8), (11), and (15) that

$$\begin{aligned}\langle k|p_C|k'\rangle^\lambda_\nu &= \frac{1}{n_p} \sum_{h \in H} \langle \nu ki|hsh^{-1}|\nu k' i\rangle^\lambda_\nu = \sum_{h,i',i''} \langle i|h|i'\rangle_\nu \times \\ &\times \langle \nu ki'|s|\nu k' i''\rangle_\lambda \langle i''|h^{-1}|i\rangle_\nu = \frac{|H|}{n_p f_\nu} \sum_i \langle \nu ki|s|\nu k' i\rangle_\lambda.\end{aligned} \tag{23}$$

Here the third equality is the result of the properties of orthogonality of the irreducib representations of the finite groups [11-13,17]. $\hat{p} \in P^{GH}$ is used to indicate the subclass consisting of elements inverse to the elements of the $p \in P^{GH}$ subclass (see [1,6]). It follows from (11) and (23) that

$$O^{\lambda\nu}_{k,k'} = \frac{f_\lambda f_\nu}{|G||H|} \sum_{p \in P^{GH}} n_p \langle k'|\hat{p}_C|k\rangle^\lambda_\nu p_C. \tag{24}$$

Placing expressions (22) for p_C in (23) and comparing the coefficients in both sides of

the acquired equality, the following is found:

$$\sum_{p\in P^{GH}} n_p \langle k|\hat{p}_C|k'\rangle_\nu^\lambda \langle k''|p_C|k'''\rangle_\nu^{\lambda'} = \delta_{\lambda\lambda'}\delta_{\nu\nu'}\delta_{k'k''}\delta_{kk'''}\frac{|G||H|}{f_\lambda f_\nu}. \tag{25}$$

The acquired relations of orthogonality (25) for the irreducible representations of the subclass algebra generalize the relations of orthogonality for the finite group [11-13,17], since the latter is the special case of (25) at H = ε.

Placing (24) in (22), $(p, q\in P^{GH})$ is found

$$\begin{aligned} p_C &= \sum_{\lambda,\nu,k,k',q} \frac{f_\lambda f_\nu n_q}{|G||H|} \langle k|p_C|k'\rangle_\nu^\lambda \langle k'|\hat{q}_C|k\rangle_\nu^\lambda q_C = \\ &= \sum_{\lambda,\nu,k,q} \frac{f_\lambda f_\nu n_q}{|G||H|} \langle k|p_C\hat{q}_C|k\rangle_\nu^\lambda q_C. \end{aligned} \tag{26}$$

Comparing the coefficients on both sides (26), it is found that

$$\sum_{\lambda,\nu,k} f_\lambda f_\nu \langle k|p_C\hat{q}_C|k\rangle_\nu^\lambda = \frac{|H||G|}{n_p}\delta_{pq}. \tag{27}$$

In the case of H = ε (27) is transformed into the known equality $\sum_\chi f_\lambda \chi^\lambda(st^{-1}) = \delta_{st}|G|$, where $\chi^\lambda(g)$ is the character of the $g\in G$ element in the irreducible representation of λ.

In the simple matrix subalgebra of the CP^{GH} algebra, assigned by the pair of indexes λ, ν, only elements such as $x\varepsilon^{\lambda\nu}$, where $x\in C$, while

$$\varepsilon^{\lambda\nu} = \sum_k O_{k,k}^{\lambda\nu} \tag{28}$$

commutate with all of the elements of the subclass algebra. In the same way, the totality of elements $\varepsilon^{\lambda\nu}$ at λ, ν, which assume all possible values, is the basis of the center of the subclass algebra. The dimensionality of the center r^C is equal to the number of λ, ν pairs for which $f_\nu^\lambda \neq 0$. It follows directly from the rules of multiplication of the elements of the subclass algebra (12) that the commutators $[O_{k,k'}^{\lambda\nu}, O_{k'',k''}^{\lambda'\nu'}]$ cover the addition of the center to CP^{GH} and, moreover, the components of these commutators in the linear space of the center are always zero. In other words, the following claim is valid: the dimensionality of the linear space covered by the commutators of the subclass algebra is equal to $(d^{GH} - r^C)$. This is the second of the two basic results (equalities III, IIIa) of Wigner's work [6] (see also [8,10]).

The primitive idempotents of the center of the group algebra are [11-13]

$$\varepsilon^\lambda = \sum_{\nu,k,l} e_{\nu kl,\nu kl}^\lambda, \tag{29}$$

where the summing is performed in terms of all ν, for which $f_\nu^\lambda \neq 0$. The CH subalgebra in CG is made up of elements such as $\sum_{h\in H} x_h h$, $x_h \in C$. For the primitive idempotent of the center of the CH subalgebra, the following is the case

$$\varepsilon^\nu = \sum_l e_{l,l}^\nu = \sum_{\lambda,k,l} e_{\nu kl,\nu kl}^\lambda, \tag{30}$$

where e_{l}^{λ}, i is the primitive idempotent of the CH algebra; here, though, the summing is performed in terms of all λ, for which $f_\nu^\lambda \neq 0$. It follows from (11), (28), (29), (30), and the rules of multiplication (7) that

$$\varepsilon^{\lambda\nu} = \varepsilon^\lambda \varepsilon^\nu. \tag{31}$$

3. ENUMERATIONS IN THE SUBMULTITUDES OF THE SUBCLASS ALGEBRA

To complete the above, it is expedient to draw a brief conclusion of the formula for the number d^G_{KH} of the double coset classes of the G group in terms of the H, K subgroups. This is even more the case due to the fact that in the literature known to the author, where this concept is encountered ([14-18], et al.), no such formulas are discovered (an incorrect expression for d^G_{KH} is given on p. 29 in [17]). The double coset class is a multitude of $KrH \subset G$, where r is one of its representations. KrH may be considered as the orbit of the subgroup K in an $H \uparrow G$ representation of the G group induced by a trivial representation of the H subgroup. One and only one trivial representation is encountered in each orbit. Therefore, d^G_{KH} is equal to the number of trivial representations of the K subgroup in $H \uparrow G$. The basis of the representation $H \uparrow G$ in the CG is $|G|/|H|$ elements

$$t_i H_C, \quad t_i \in G_\pi, \tag{32}$$

where $H_C = \sum_{h \in H} h$ and G_π is a system of representations of the left coset G classes in terms of H. The element $s \in G$ in $H \uparrow G$ is represented by the formulation

$$s : t_i H_C \mapsto s t_i H_C, \quad t_i \in G_\pi. \tag{33}$$

The character $\chi_{GH}(s)$ of the element $s \in G$ is found in an $H \uparrow G$ representation. As a result of (33), it is equal to the number of such t_i that $t_i^{-1} s t_i \in H$, but with such t_i and $(t_i h)^{-1} s (t_i h) \in H$ for any $h \in H$. Therefore, the product of $|H| \chi_{GH}(s)$ shows how many times in the totality of elements of tst^{-1} at t which traverses the values of all elements of the group the elements of the H subgroup are encountered. This totality is a class of adjoined elements (assume that this is the ρ class), and each of the encountered elements appears $|G|/g_\rho$ times. Consequently, for any element of this class, $s \in \rho$, the following is found:

$$\chi_{GH}(s) = \frac{|G|}{|H|} \frac{h_\rho}{g_\rho}. \tag{34}$$

The equality (4) is found from (34), after using the relations of the orthogonality of the irreducible characters for finding the frequency of the trivial representation of the K subgroup in $H \uparrow G$.

Assume that $\bar{\nu}$ is the trivial representation of the subgroup H. Each double coset class L = HrH is a combination of a certain number of subclasses (since $hHrHh^{-1}$ = HrH for each $h \in H$). Assume that

$$l_C = \sum_{t \in L} t. \tag{35}$$

The elements of the CG, such as (35), make up the subalgebra in the subclass algebra (the CD^{GH} algebra of the dual coset classes). The $e \in CP^{GH}$ element belongs to the CD^{GH} algebra if and only if

$$\bar{H}_C e = e \bar{H}_C = e, \tag{36}$$

where $\bar{H}_C = H_C / |H| = e^{\bar{\nu}} = e^{\bar{\nu}}_{i,i}$ (i here assumes only one value). In the basis (11) the elements $e^{\lambda\bar{\nu}}_{k,k'}$ and only they have the property (36). In the same way their number is equal to the number of double coset classes of the G group in terms of the H subgroup (i.e., the dimensionality of the CD^{GH} algebra). Thus, the following enumerating result is found (compare (4)):

$$\sum_\lambda (f^\lambda_{\bar{\nu}})^2 = d^G_{HH} = \frac{|G|}{|H|^2} \sum_\rho \frac{(h_\rho)^2}{g_\rho}. \tag{37}$$

Now the author finds how many subclasses are contained in the class of adjoined elements α of the G group. The author examines the transitive representation of G through conjugation for $\alpha : s \in G$, which is represented in it by the formulation

$$s: t \mapsto sts^{-1}, \quad t \in \alpha. \tag{38}$$

The representation (38) is isomorphouse to $N_\alpha \uparrow G$, where N_α is the stabilizer of one (any, but a fixed one) element of the α class (compare (13) and (14) at H = G). The subclass is an orbit of the H subgroup in the basis of the (38) representation. But the number of the orbits and in the same way the number d_α^{GH} of the subclasses in the α class as a result of the information presented at the beginning of the section is equal to the number of double coset classes of the G group in terms of the H and N_α subgroups (see (4))

$$d_\alpha^{GH} = d_{N_\alpha H}^{G} = \frac{|G|}{|N_\alpha||H|} \sum_\rho \frac{h_\rho n_{\rho\alpha}}{g_\rho}, \tag{39}$$

where $n_{\rho\alpha}$ is the number of elements of the N_α subgroup in the class of adjoined elements θ of the G group.

The author finds the full number of d^{GH} subclasses. The following is true:

$$|G| \frac{n_{\rho\alpha}}{|N_\alpha|} = n_{\rho\alpha} g_\alpha = Q_{\rho\alpha} = Q_{\alpha\rho} = n_{\alpha\rho} g_\rho = |G| \frac{n_{\alpha\rho}}{|N_\rho|}, \tag{40}$$

where $Q_{\rho\alpha}$ is the number of ordered pairs of commutating elements of the group, one of which belongs to the α class, while the other belongs to the ρ class, i.e.,

$$Q_{\rho\alpha} = |\{\langle s, t\rangle | st = ts, \ s \in \alpha, \ t \in \rho\}|. \tag{41}$$

The symmetry of $Q_{\rho\alpha}$ with respect to the transposition of the indexes follows from (41), while the other equalities in (40) are the direct results of definitions. Since $\sum_\alpha n_{\alpha\rho} = |N_\rho|$, it follows from (40) that

$$\sum_\alpha \frac{n_{\rho\alpha}}{|N_\alpha|} = \sum_\alpha \frac{n_{\alpha\rho}}{|N_\rho|} = 1. \tag{42}$$

Having summed (39) in terms of α and after using (42), formula (3) is found for the number of all subclasses.

The authors find the dimensionality r_λ of the subalgebra of the CP^{GH} algebra, which consists of elements such as $\varepsilon^\lambda p_C, \ p \in P$. On one hand, as a result of (11), (12), and (29)

$$r_\lambda = \sum_\nu (f_\nu^\lambda)^2. \tag{43}$$

However, the author uses the approach which was already used in deducing (39). Representation λ will be assumed to be assigned in any basis, and the following transform at $s \in G$ is examined:

$$s: e_{a,b}^\lambda \mapsto s e_{a,b}^\lambda s^{-1} = \sum_{a',b'} \langle a'|s|a\rangle_\lambda \langle b|s^{-1}|b'\rangle_\lambda e_{a',b'}^\lambda. \tag{44}$$

Here the equality is a result of (7), (8). The transforms (44) assign the representation $\lambda \times \tilde{\lambda}$ of the G group, which are the direct product of representation λ and $\tilde{\lambda}$ representation which is contragradient to it. The character of the $s \in \rho$ element in $\lambda \times \tilde{\lambda}$ is equal to $\chi_\rho^\lambda (\chi_\rho^\lambda)^*$, where χ_ρ^λ and $(\chi_\rho^\lambda)^*$ are the characters of the class ρ in the λ and $\tilde{\lambda}$ representations, respectively; the sign * indicates complex conjugation. The above follows from (44) since, assuming $a' = a$ and b' = b and summing, the following is found for the character of representation (44):

$$\begin{aligned} \mu_\rho &= \sum_{a,b} \langle a|s|a\rangle_\lambda \langle b|s^{-1}|b\rangle_\lambda = \\ &= \Big(\sum_a \langle a|s|a\rangle_\lambda\Big)\Big(\sum_b \langle b|s|b\rangle_{\tilde{\lambda}}^*\Big) = \chi_\rho^\lambda (\chi_\rho^\lambda)^*. \end{aligned} \tag{45}$$

The following is true in the basis (10):

$$\sum_{h\in H} h e^{\lambda}_{ii,\nu k' i'} h^{-1} = \sum_{h, i'', i'''} \langle i''|h|i\rangle_\nu \langle i'|h^{-1}|i'''\rangle_\nu e^{\lambda}_{ii'',\nu k' i'''} =$$
$$= \sum_{i'', i'''} \delta_{\nu\nu'} \delta_{ii'} \delta_{i'' i'''} \frac{|H|}{f_\nu} e^{\lambda}_{iii'',\nu k' i'''} = \frac{|H|}{f_\nu} \delta_{\nu\nu'} \delta_{ii'} O^{\lambda\nu}_{k,k'}. \tag{46}$$

Hence it follows that the unknown number r_λ is equal to the frequency of appearance of the trivial representation of the H subgroup in the (44) representation, i.e., in $\lambda \times \tilde{\lambda}$. The only remaining thing is to use the properties of orthogonality of the characters

$$r_\lambda = \frac{1}{|H|} \sum_p |\chi^\lambda_p|^2 h_p = \sum_\nu (f^\lambda_\nu)^2. \tag{47}$$

Here $|z|$ is the modulus of the complex number z.

Summing (47) in terms of λ and using the properties of orthogonality of the columns in the tables of the irreducible characters of the finite groups, the second proof of equality (3) is acquired.

4. AMBIVALENT SUBCLASSES

The class of adjoined elements of a group which contain s^{-1} along which each s is called ambivalent. It is natural that this term is used to characterize the subclasses which have an analogous property. The following is true for the ambivalent subclass:

$$p = \hat{p}, \quad p_C = \hat{p}_C. \tag{48}$$

Since $(st)^{-1} = t^{-1}s^{-1}$, then

$$(\widehat{p_C l_C}) = \widehat{l_C}\,\widehat{p_C}, \quad p, l \in P^{GH}. \tag{49}$$

It follows from (48), (49) that when all subclasses are ambivalent, then the algebra of the subclasses is commutative. But as a result of proposition 1 the commutativism means that the dimensionality $(f^\lambda_\nu)^2$ of simple matrix subalgebras of the CP^{GH} algebra does not exceed one. Therefore, the following claim is valid [1-4]: if all of the subclasses of the G group are ambivalent with respect to the H subgroup, then the frequencies of representations of the subgroup in the limitation on their irreducible representations of the group do not exceed one. The condition here is sufficient, but not necessary.

The author finds how many ambivalent subclasses are contained in the multitude of subclasses P^{GH}. The following is introduced into examination over and above the already examined transformations in the group algebra (compare (44)) for this purpose:

$$s: t \mapsto sts^{-1}, \quad s, t \in G, \tag{50}$$

and the following as well:

$$I: t \mapsto t^{-1}, \quad t \in G. \tag{51}$$

It is directly tested that the transformations (50) and (51) commutate and in the same way in totality are isomorphous to the direct product of the $G \times S_2$ group, where $S_2 = \{\varepsilon, I\}$ group, which consists of two elements. It was already noted that the subclass is the orbit of the H subgroup in the representation (50) or the basis of the trivial representation $\tilde{\nu}$ of the H subgroup in the CG group algebra (compare (16), (46)), acquired by limiting representation (50) to H. In accordance with the noted commutation, transformation (51) does not emerge from the subclass algebra:

$$I: p_C \mapsto \hat{p}_C, \quad p_C, \hat{p}_C \in CP^{GH}. \tag{52}$$

Reduction with respect to S_2 is quite simple

$$(p_C + \hat{p}_C), \quad p_C, \hat{p}_C \in CP^{GH}, \tag{53a}$$

is the basis of the symmetrical representation [2], while

$$(p_C - \hat{p}_C), \quad p_C, \hat{p}_C \in CP^{GH}, \tag{53b}$$

is the basis of the antisymmetrical representation $[1^2]$. It follows from (53a, 53b) that the number of ambivalent subclasses d_α^{GH} is equal to the difference in the frequencies of the representations [2] and $[1^2]$ of the S_2 group in the basis of the subclass algebra. Or it is equal to the difference in the frequencies of the representations of $\bar{v}\times[2]$ and $\bar{v}\times[1^2]$ of the $H \times S_2$ subgroup in the representation (50), (51) in the basis of the CG group algebra. The above in terms of the characters of the groups means the following:

$$d_\alpha^{GH}=\frac{1}{2|H|}\sum_{h\in H}\left(\gamma(h)+\gamma(I\times h)\right)-\frac{1}{2|H|}\sum_{h\in H}\left(\gamma(h)-\gamma(I\times h)\right)=$$
$$=\frac{1}{|H|}\sum_{h\in H}\gamma(I\times h), \tag{54}$$

where $\gamma(h)$ and $\gamma(I \times h)$ are the characters of the corresponding elements in the limitation of (50), (51) group $G \times S_2$ on the subgroup $H \times S_2$.

$\gamma(I \times h)$ is found. In the basis of the relative regular representation of the G group (6) the transformation of I has the following appearance:

$$I: e^\lambda_{a,b}\mapsto(e^\lambda_{b,a})^*. \tag{55}$$

(55) is the direct consequence of (6), (51) and the Schur-Auerbach theorem [17] about the unitary state of the irreducible representations of the finite groups (the sense of the * designation here is unambiguous). When the representation λ and that of $\tilde{\lambda}$ contragradient to it are not equivalent, then, since (55) means a shift from one to the other, the space of the dimensionality $2(f_\lambda)^2$, assigned by λ and $\tilde{\lambda}$, introduces a zero contribution to $\gamma(I \times h)$, since the two spaces are invariant with respect to H. Such representations are called third order representations [17,19]. But when the representations λ and $\tilde{\lambda}$ are equivalent, then there is a unitary matrix $||u_{a,b}||$, which transforms one representation into the other and, therefore, for such λ (see (6))

$$(e^\lambda_{b,a})^*=\sum_{a',b'}u_{a,a'}e^\lambda_{b',a'}u^*_{b,b'}. \tag{56}$$

In this case (see [19]):

$$u_{a,a'}=c_\lambda u_{a',a}, \tag{57}$$

where $c_\lambda = 1$ for first order representations (the matrices of the representations are real) and with $c_\lambda = -1$ for second order representations (the characters are real, but the representations are not equivalent to the real ones). For representations of these two orders, the following is true as a result of (7), (8), (56):

$$I\times h: e^\lambda_{a,b}\mapsto h(e^\lambda_{b,a})^*h^{-1}=$$
$$=\sum_{a',b',a'',b''}\langle a''|h|b'\rangle_\lambda u_{a,a'}u^*_{b,b'}\langle a'|h^{-1}|b''\rangle_\lambda e^\lambda_{a'',b''}. \tag{58}$$

Having assumed $b'' = b$ and $a'' = a$, having summed the diagonal coefficients of transformation (58) in terms of a, b, and having used (57), the following is found for the character in space assigned by the irreducible representation λ of the discussed two orders:

$$\gamma^\lambda(I\times h)=c_\lambda\chi^\lambda(h^2). \tag{59}$$

Thus, assuming $c_\lambda = 0$ for representations of the third order, the following is found for the full character:

$$\gamma(I\times h)=\sum_\lambda c_\lambda\chi^\lambda(h^2). \tag{60}$$

As follows from the results of Frobenius (see [19]), the sum in the right side of (60) is equal to the number $r_2(h)$ solutions in the group of equation $s^2=h^2, s\in G$. Thus, ex-

pression (5) is found from (54) and (60) for the number of ambivalent subclasses d_{α}^{GH}.

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