Symmetric Group Representations and Noncanonical Bases in the Subduction $U_{k(k+1)/2} \supset U_k$

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Fifteen years ago canonical bases of the irreducible unitary group representations (the Gel'fand bases) were constructed with the aid of symmetric group representation theory.[1] These are orthonormal bases in tensor space of the representations reduced with respect to the subgroup sequence $U_k \supset U_{k-1} \supset \cdots \supset U_1$. The method was used to obtain expressions for the Clebsch–Gordan coefficients and transformation matrices of unitary groups for some of multiplicity-free cases.[1–4] The approach has not gotten much attention since, but the results of Refs. 1 and 4 are now being independently rediscovered by different authors. The purpose of this paper is to describe how to construct along the same lines, i.e., with the technique of symmetric group theory, the noncanonical bases of unitary groups in the subduction $U_{k(k+1)/2} \supset U_k \supset U_{k-1} \supset \cdots \supset U_1$. It must be pointed out that in the case of $k=2$ this gives the bases for the reduction $U_3 \supset O_3$ due to the homomorphism $U_2 \rightarrow O_3$; this fact seems to have been unnoticed until now. Because of the reduction $U_{3k} \supset U_k \times U_3 \supset S_k \times O_3$ it follows that a considerable part of the mathematics of the nuclear shell model with oscillator potential is a part of symmetric group representation theory.[5,6] As another application let us mention the Interacting Boson model;[7] in this case $k=3$.

The two bases mentioned (canonical and noncanonical) of irreducible representations of $U_{k(k+1)/2}$ in the tensor space will be constructed, then the transformation matrix between the two bases will be expressed in terms of the matrix which reduces the representation of the symmetric group induced by the representation of its subgroup, the subgroup itself being the wreath product of two symmetric groups. We begin with restating some results of,[1] thus introducing the notation and making the transition to noncanonical bases of $U_{k(k+1)/2}$ straightforward.

The r-dimensional unitary matrix $\|u(ij)\|$ is called the fundamental representation, denoted $\{1\}$, of the group U_r. The nth rank tensor is the basis of nth Kronecker power of the representation $\{1\}$. Define the action of permutation $s \in S_n$ on the tensor component by†

$$sF(i_1, i_2, \ldots, i_n) = F(i_{s^{-1}(1)}, i_{s^{-1}(2)}, \ldots, i_{s^{-1}(n)}).$$

Let

$$O^{\lambda}_{\rho'\rho} = \frac{f_\lambda}{n!} \sum_{s \in S_n} \langle \rho | s^{-1} | \rho' \rangle s$$

be the Young operator of orthogonal Young representation, i.e., of the orthogonal representation reduced with respect to the subgroup sequence $S_n \supset S_{n-1} \supset \cdots \supset S_1$. Here λ stands for Young diagram $\{\lambda\}$, indicating the irreducible representation of dimension f_λ, ρ' and ρ are standard Young tableaux of diagram $\{\lambda\}$, $\langle \rho | s^{-1} | \rho' \rangle$ is the matrix element of s^{-1} in the irreducible representation. Let m_l $(l = 1, 2, \ldots, r)$ be the number of occurrences of vector component l in $F(i_1, i_2, \ldots, i_n)$. The vector $m = (m_1, m_2, \ldots, m_r)$ is called the weight vector. Choose among the tensor components of weight m the one with lexicographic order of indices. Denote it by $\bar{F}_m$. Let P_m be the sum of the elements of the group $S(m)$ of permutations of equal indices in $\bar{F}_m$. Because of the equality $\langle \rho | P_m | \rho'' \rangle = d^m_\rho d^m_{\rho''}$ we have

$$O^{\lambda}_{\rho'\rho} \bar{F}_m = O^{\lambda}_{\rho'\rho} \left(\prod_{l=1}^{r} m_l! \right)^{-1} P_m \bar{F}_m = \sum_{\rho'' \vdash \bar{\rho}(m)} O^{\lambda}_{\rho'\rho''} d^m_\rho d^m_{\rho''} \bar{F}_m, \tag{1}$$

where the summation is taken with respect to all standard tableaux obtained from ρ by permutations from $S(m)$. In Eq. (1)

$$d^m_\rho = \prod_{p=1}^{r} \prod_{\Sigma_{l=1}^{p-1} m_l < i < j \leqq \Sigma_{l=1}^{p} m_l} \left(1 + \frac{1}{x_j(\rho) - x_i(\rho)} \right)^{1/2} \tag{2}$$

†This definition is more convenient than the one used in Ref. 1.

where $x_j(\rho)$ is equal to the column number minus row number of the box with symbol j in the standard Young tableau ρ. The symbol $\bar{\rho}(m)$ in Eq. (1) denotes the generalized Young tableau which is obtained when symbols i, satisfying the inequality $\Sigma_{l=1}^{p-1} m_l < i \leqq \Sigma_{l=1}^{p} m_l$, are replaced in ρ by p. The meaning of notation $\rho \vdash \bar{\rho}(m)$ thus becomes clear. Let us give an example:

$$m=(1,2,4,0,2),\quad \rho \vdash \bar{\rho}(m)\colon \rho \Leftrightarrow \begin{array}{ccccc} 1 & 2 & 4 & 5 & 6 \\ 3 & 7 & 8 & & \\ 9 & & & & \end{array},\quad \bar{\rho}(m) \Leftrightarrow \begin{array}{ccccc} 1 & 2 & 3 & 3 & 3 \\ 2 & 3 & 5 & & \\ 5 & & & & \end{array}$$

From Eqs. (1) and (2) it follows that (1) vanishes if $\bar{\rho}(m)$ has two equal symbols in the same column. It is assumed that the generalized Young tableau has no such pair of symbols. The expressions

$$\frac{1}{d^m_{\rho(m)}} \sqrt{\frac{n!}{f_\lambda}}\, O^\lambda_{\rho'\rho(m)} \bar{F}_m = \left(\prod_{l=1}^{r} m_l!\right)^{-1} \sqrt{\frac{n!}{f_\lambda}} \sum_{\rho \vdash \bar{\rho}(m)} d^m_\rho O^\lambda_{\rho'\rho} \bar{F}_m \tag{3}$$

are the elements of the orthonormal canonical basis of the irreducible representation $\{\lambda\}$ of U_r. The left-hand side of Eq. (3) is Eq. (2.9) in Ref. 1 and the equality follows from Eq. (1). $\rho(m)$ is arbitrary but of fixed tableau $\rho(m) \vdash \bar{\rho}(m)$. We have f_λ equivalent bases, numbered by ρ'. $\bar{\rho}(m)$ defines the Gel'fand–Tsetlin diagram as follows: the pth row $\{m_{p1}, m_{p2}, \ldots, m_{pp}\}$ in the Gel'fand–Tsetlin diagram is the Young diagram obtained from $\bar{\rho}(m)$ by deleting all the boxes with symbols exceeding p. This diagram indicates the irreducible representation of subgroup $U_p \subset U_r$.

Now we are in a position to discuss the case $U_{k(k+1)/2} \supset U_k$. Let us embed U_k in $U_{k(k+1)/2}$ as follows: as a representative of $\|u(ij)\| \in U_k$ $(i, j = 1, 2, \ldots, k)$ choose the matrix

$$\|[1+\delta(j_1 j_{1'})]^{-1/2}[1+\delta(i_1 i_{1'})]^{-1/2}\{u(j_1 i_1)u(j_{1'} i_{1'}) + u(j_{1'} i_1)u(j_1 i_{1'})\}\| \tag{4}$$

from $U_{k(k+1)/2}$ [$\delta(ij)$ is the Kronecker symbol]. Clearly, this is the representation $\{2\}$ of U_k with the basis

$$2^{-[1+\delta(i_1 i_{1'})]/2}\{F(i_1, i_{1'}) + F(i_{1'}, i_1)\}. \tag{5}$$

$U_{k(k+1)/2}$ is the group of arbitrary unitary transformations in the space (5), while when restricted to U_k, (5) transforms according to (4). Consider now the Cartesian nth power of (5). Under U_k group action it transforms like the normalized tensor of rank $2n$ with components

$$2^{-(r+n)/2} P_2^n F(i_1, i_{1'}, i_2, i_{2'}, \ldots, i_n, i_{n'}), \tag{6}$$

where P_2^n is the sum of all the permutations within the pairs $\{p, p'\}$ $(p=1,2,\ldots,n)$ (denote the corresponding group by S_2^n), while under the $U_{k(k+1)/2}$ group action it transforms as an nth rank tensor. In what follows let us understand S_n as the group of pairs $\langle s, s'\rangle$ where s' and s permute, respectively, the sets of stroked and unstroked symbols, but both the sets in the same way. Let S_{2n} be the group of all the permutations of subscripts of Eq. (6). The products $q \circ \langle s, s'\rangle$ with $\langle s, s'\rangle \in S_n$, $q \in S_2^n$ form a subgroup of S_{2n} called the wreath product of S_2 and S_n, denoted $S_2 \operatorname{Wr} S_n$. It is isomorphic to the n-dimensional hyperoctahedral group.[8] Let $\mu_{ij} = \mu_{ji}$ be equal to the number of pairs $\{i_p, i_{p'}\} = \{i, j\}$ in Eq. (6). Clearly, $r = \sum_{i=1}^k \mu_{ii}$. Let $\mu_i = \sum_{i=1}^k \mu_{ij}[1+\delta(ij)]$. The set $m = \{\mu_{ij}\}$ is, of course, the weight vector of $U_{k(k+1)/2}$. We may assume the set of pairs $\{i, j\}$ to be arbitrarily numbered by the symbols $1, 2, \ldots, k(k+1)/2$ and apply the procedure described above to obtain the canonical basis of $U_{k(k+1)/2}$. But the representation will be, in general, reducible when subduced on U_k. Let $\bar{F}_\mu$ be again the component with lexicographic order of indices $1, 2, \ldots, k$, the ith one occurring μ_i times. Choose permutations p_m as those representatives of double cosets (one from each) with respect to subgroups $S_2 \operatorname{Wr} S_n$ and $S(\mu)$ which send $\bar{F}_\mu$ into

$$\bar{F}_m = 2^{-(r+n)/2} P_2^n p_m \bar{F}_\mu, \tag{7}$$

i.e., $\bar{F}_m$ has lexicographic order of pairs of indices (in the numbering chosen). The weight $\{\mu_{ij}\} = m$ of $U_{k(k+1)/2}$ is uniquely determined by the weight $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of U_k and by the representative p_m. In analogy with Eq. (3) we obtain the canonical basis

$$2^{-(r+n)/2} \left(\prod_{l=1}^k \mu_l! \right)^{-1} \frac{1}{d_{\rho(m)}^m} \sqrt{\frac{n!}{f_\lambda}}\, O^\lambda_{\rho' \rho(m)} P_n^2 p_m P_\mu \bar{F}_\mu. \tag{8}$$

Here we must point out that multiplication of Eq. (8) on the left by $O^{\lambda}_{\rho'\rho'}$ will not change the expression. Let $\|\langle\rho'_2|\alpha(\lambda_2,\lambda),\rho'\rangle\|$ be the matrix which reduces the orthogonal Young representation with respect to the subgroup $S_2\,\mathrm{Wr}\,S_n$. (Here $\alpha(\lambda_2,\lambda)$ labels repeated representations $\{\lambda\}$ of $S_2\,\mathrm{Wr}\,S_n$ in the representation $\{\lambda_2\}$ of S_{2n}.) From Eq. (3) it is easy to see that

$$\left(\prod_{l=1}^{k}\mu_l!\right)^{-1}\sqrt{\frac{(2n)!}{f_{\lambda_2}}}\sum_{\rho'_2}\langle\rho'_2|\alpha(\lambda_2,\lambda),\rho'\rangle\sum_{\rho_2\vdash\bar{\rho}_2(\mu)} d^{\mu}_{\rho_2}O^{\lambda_2}_{\rho'_2\rho_2}P_\mu\bar{F}_\mu \tag{9}$$

is the orthonormal basis of the representation of $U_{k(k+1)/2}$ reduced with respect to the subgroup U_k. It has the above mentioned property regarding left multiplication by $O^{\lambda}_{\rho'\rho'}$.

Consider the case $\mu_i=1$ for all $i=1,2,\ldots,k$ $(k\geqq 2n)$, then $P_\mu=\varepsilon$, $\rho(m)=\rho$. If we delete $\bar{F}_\mu$ from Eqs. (8) and (9) and multiply the remaining expressions on the right by $s\in S_{2n}$, it transforms as a representation of S_{2n}, induced by the corresponding irreducible representation of $S_2\,\mathrm{Wr}\,S_n$. ($2^{-n}O^{\lambda}_{\rho'\rho}P^2_n$ are Young operators of this representation.) In basis (8) the representation is reducible, while in (9) it is reduced. If $\|\langle p_m,\rho|\alpha(\lambda_2,\lambda),\lambda_2,\rho_2\rangle\|$ is the matrix, reducing the representation then expanding (8) in terms of (9) we obtain for the coefficients the expression

$$\frac{1}{2^{r/2}d^{m}_{\rho(m)}}\sum_{\rho_2\vdash\bar{\rho}_2(\mu)}\langle p_m,\rho|\alpha(\lambda_2,\lambda),\lambda_2,\rho_2\rangle\, d^{\mu}_{\rho_2}. \tag{10}$$

In conclusion let us remark that the procedure described might be generalized for the reduction $U_{(k+q-1)!/q!(k-1)!}\supset U_k$ (in our case $q=2$).

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