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The bijection between plane partitions and nonnegative integer matrices

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Abstract. The one-to-one correspondence between the set of plane partitions with r rows and m columns and the set of matrices of nonnegative integers with the same numbers of rows and columns has been constructed.

1. Restricted plane partition is the matrix $||a_{ij}||$ with r rows and m columns the elements of which turn to be nonnegative integers, satisfying:

$$a_{ij} \geqslant a_{ik} \quad \text{for} \quad 1 \leqslant j < k \leqslant m, \quad i = 1, \ldots, r, \tag{1}$$

$$a_{ji} \geqslant a_{ki} \quad \text{for} \quad 1 \leqslant j < k \leqslant r, \quad i = 1, \ldots, m. \tag{2}$$

It means that the matrix elements are nonincreasing from the left upper corner both in the rows and in the columns. Much information on plane partitions can be found in [1–5], especially regarding the generating functions for the numbers of distinct plane partitions with various restrictions imposed, additional as well, or some restrictions deleted (e.g., the finiteness of r or m).

In the present note the one-to-one correspondence between the set of restricted plane partitions and the set of nonnegative integer matrices $||b_{ij}||$ with r rows and m columns has been constructed. Let us give an example; the following two matrices are the images of each other in the bijection ($r = m = 4$):

$$||a_{ij}|| = \begin{Vmatrix} 10 & 10 & 9 & 4 \\ 9 & 5 & 3 & 2 \\ 7 & 3 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{Vmatrix}, \qquad ||b_{ij}|| = \begin{Vmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{Vmatrix}. \tag{3}$$

It would be interesting to compars the correspondence presented here with analogous ones [5–8] based on the use of Schensted algorithm [9].

In the course of the construction of the bijection we prove the following theorem on generating function, the some of the known ones being the special cases of the theorem.

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THEOREM. *In the expansion*

$$\prod_{i=1}^{r}\prod_{j=1}^{m}\left(1-x^{i+j-1}\prod_{l=1-i}^{j-1} y_l\right)^{-1}$$
$$=\sum_{k(-r+1)=0}^{\infty}\sum_{k(-r+2)=0}^{\infty}\cdots\sum_{k(m-1)=0}^{\infty} C(k(-r+1),k(-r+2),\tag{4}$$
$$\ldots,k(m-1))x^k\prod_{l=-r+1}^{m-1} y_l^{k(l)}$$

the coefficient $C(k(-r+1),k(-r+2),\ldots,k(m-1))$ *is equal to the number of restricted plane partitions with the sums*

$$\sum_{j=i-l} a_{ij}=k(l)\qquad\left(k=\sum_{l=-r+1}^{m-1} k(l)=\sum_{i=1}^{r}\sum_{j=1}^{m} a_{ij}\right).\tag{5}$$

In the bijection for the corresponding images $||b_{ij}||$ *of* $||a_{ij}||$ *for nonnegative* l:

$$\sum_{i=1}^{r}\sum_{j=l+1}^{m} b_{ij}=k(l),\tag{6}$$

and for nonpositive l:

$$\sum_{i=-l+1}^{r}\sum_{j=1}^{m} b_{ij}=k(l).\tag{7}$$

Assuming in (4) $y_l=1$ for all l and letting $m\to\infty$ we obtain the well known result of MacMahon [2] on generating function for the number of plane partitions with r rows (eq. (11.2.14) in [1]). On the other hand, let $m\to\infty$ and let us replace each row of $||a_{ij}||$, itself being a linear partition; by the conjugate partition. We thus obtain that the restriction of finiteness of m may be replaced by the requirement that the greatest part of the plane r-rowed partition must not exceed m. Assuming then in (4) for all $l\neq 0$, $y_l=1$, we achieve one of the results of the work [4] letting $y_0=q$.

Now let us proceed with the actual construction of the bijection.

2. The bijection is constructed recursively. In the construction of the bijection two notions are essential: the *insertion algorithm* and the *order of application* of the insertion algorithm. In the process of the application of the insertion algorithm the value of some matrix element b_{st} is decreased by one while the sum $\sum_{i,j} a_{ij}$ is increased by $(s+t-1)$ because the values of such a number of matrix elements of $||a_{ij}||$ are increased by one. The process begins with the matrix $||b_{ij}||$ and the zero plane partition and continues until $||b_{ij}||$ is zeroed and simultaneously the final restricted plane partition $||a_{ij}||$ is constructed. Let us describe this thoroughly.

Order of application of the insertion algorithm is the following:

$$b_{1m}, b_{1m-1}, \ldots, b_{11}, b_{2m}, b_{2m-1}, \ldots, b_{21}, b_{3m}, b_{3m-1}, \ldots, b_{r1}. \tag{8}$$

In the first step of the application the value of the first (from the left) nonzero matrix element in (8) is decreased by one. After n applications of the insertion algorithm the sequence (8) is replaced by the sequences

$$b'_{1m}, b'_{1m-1}, \ldots, b'_{11}, b'_{2m}, b'_{2m-1}, \ldots, b'_{21}, b'_{3m}, b'_{3m-1}, \ldots, b'_{r1}. \tag{8'}$$

Obviously here $\sum_{i,j} b'_{ij} = \sum_{i,j} b_{ij} - n$. In the $(n+1)$-th step of the application of the insertion algorithm the value of the first nonzero element of the sequence (8′) is decreased by one. And so on until all the elements b_{ij} of (8) are zeroed.

The insertion algorithm begins with the plane partition all matrix elements (parts) of which are zero. Let b_{1v} be the first nonzero element in (8) (the rows are numbered from the first nonzero one). The first step of the algorithm decreases b_{1v} by one and istead of zero plane partition constructs the plane partition with v once in the first row:

$$\|\overbrace{111\ldots1}^{v}0\ldots0\|. \tag{9}$$

After n applications of the insertion algorithm the plane partition $\|a'_{ij}\|$ is constructed. Now let us describe the $(n+1)$-th application of the insertion algorithm to $\|a'_{ij}\|$ to give as the result the plane partition $\|a'_{ij}\|$. Let b_{st} be the first nonzero element in (8). Then begin by increasing a'_{1t} by one, i.e.,

$$a''_{1t} = a'_{1t} + 1. \tag{10}$$

If $a'_{1t-1} = a'_{1t}$, then again

$$a''_{1t-1} = a'_{1t-1} + 1. \tag{10'}$$

And it proceeds until for some τ $a'_{1\tau-1} > a'_{1\tau}$. Then increase $a'_{2\tau}$ by one, i.e.,

$$a''_{2\tau} = a'_{2\tau} + 1. \tag{11}$$

If $a'_{2\tau-1} = a'_{2\tau}$, then again

$$a''_{2\tau-1} = a'_{1\tau-1} + 1. \tag{11'}$$

And it proceeds until for some φ $a'_{2\varphi-1} > a'_{2\varphi}$. Then go to the third row and increase $a'_{3\varphi}$ by one... And so on until the $(n+1)$-th step of the insertion algorithm *terminates* by increasing by one the first element of the s-th row:

$$a''_{s1} = a'_{s1} + 1. \tag{12}$$

Note. For the sake of rigorousity it must be assumed that for all i

$$a_{i0} = a'_{i0} = a''_{i0} = \infty. \tag{13}$$

In other words, when the element a'_{i1} of the first column is increased by one, the insertion algorithm proceeds with the first element of subsequent row of the plane partition.

Example. For the matrix $\|b_{ij}\|$ (3) the sequence (8) is

$$1,2,0,1,2,0,1,2,1,3,0,0,0,2,1,0.$$

The plane partition (9) is:

$$\|1111\|.$$

After $n = 11$, application of the insertion algorithm, the sequence (8′) is

$$0,0,0,0,0,0,0,0,0,2,0,0,0,2,1,0 \tag{8′e}$$

and the plane partition $\|a'_{ij}\|$ is:

$$\|a'_{ij}\| = \begin{Vmatrix} 6 & 5 & 5 & 4 \\ 5 & 4 & 3 & 2 \\ 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \tag{14}$$

12-th step of the insertion algorithm, applied to (14), constructs the plane partition

$$\|a''_{ij}\| = \begin{Vmatrix} 6 & 6 & 6 & 4 \\ 5 & 5 & 3 & 2 \\ 3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \tag{15}$$

and replaces first nonzero number 2 in (8′e) by 1.

3. Let us prove that the correspondence constructed is one-to-one or, in other words, that the function F: $\|b_{ij}\| \to \|a_{ij}\|$ is bijective.

PROPOSITION 1. *The function F: $\|b_{ij}\| \to \|a_{ij}\|$ constructed is injective.*

Proof. It must be shown that F maps two different matrices onto two different plane partitions. The proof is by induction due to recursive character of the algorithm. For the matrices $\|b_{ij}\|$ with only the first row nonzero the proposition is true because the corresponding plane partition constructed by the algorithm is simply linear [1] (one-row) partition:

$$\left\| \sum_{i=1}^{m} b_{1i},\ \sum_{i=2}^{m} b_{1i},\ \sum_{i=3}^{m} b_{1i}, \ldots, b_{1m-1} + b_{1m},\ b_{1m} \right\|. \tag{16}$$

(In the case of all the $b_{1i} = 0$ the statement remains true as well.) Let the Proposition 1 be true for the plane partitions, constructed by n steps of application of the insertion algorithm. Then for the two sequences (8′) with the same elements preceeding the ome in the st-position and, maybe, differing by $b'_{st} \neq \tilde{b}'_{st}$, the two plane partitions constructed by $(n+1)$-th application of the insertion algorithm equals each other if and only if $b'_{st} > 0$ and $\tilde{b}'_{st} > 0$. This statement completes the proof.

PROPOSITION 2. *The function* F: $\|b_{ij}\| \to \|a_{ij}\|$ *constructed is surjective.*

Proof. We must show that set of matrices $\|b_{ij}\|$ is mapped by F onto the set of all restricted plane partitions. The proof is by the construction and the use of generating functions with the simultaneous proof of the first part (eq. (4), (5)) of the theorem formulated in the first section. Let us introduce the notion of the coordinate of the matrix elements a_{ij} and b_{ij} of the matrices $\|a_{ij}\|$ and $\|b_{ij}\|$ we are working with as being equal to $y_{(j-i)} \equiv y_{j-i}$. Thus we have $(r+m-1)$ different coordinates $y_{-r+1}, y_{-r+2}, \ldots, y_0, \ldots, y_{m-1}$, the coordinates of all the matrix elements on the same diagonal being equal (thus, y_0 is the coordinate of all the matrix elements on the leading diagonal). With each restricted plane partition $\|a_{ij}\|$ let us associate the product

$$\prod_{i=1}^{r}\prod_{j=1}^{m}(y_{j-i})^{a_{ij}} \equiv f(\|a_{ij}\|) \tag{17}$$

and let us assume in what follows that the coordinates are complex variables satisfying $|y_l| < 1$ for all indices l. It follows from the description of the insertion algorithm and from (17) that if b_{st} applications of it from $(n+1)$-th step to $(n+b)$-th one constructs the plane partition $\|a'''_{ij}\|$ from the plane partition $\|a'_{ij}\|$, then

$$f(\|a'''_{ij}\|) = f(\|a'_{ij}\|)\left(\prod_{l=-s+1}^{t-1} y_l\right)^{b_{st}}. \tag{18}$$

From (18) and the description of the insertion algorithm we conclude that

$$\begin{aligned}
&\prod_{i=1}^{r}\prod_{j=1}^{m}\left(1-\prod_{l=-i+1}^{j-1} y_l\right)^{-1} \\
&= \sum_{k(-r+1)=0}^{\infty}\sum_{k(-r+2)=0}^{\infty}\cdots\sum_{k(m-1)=0}^{\infty} C(k(-r+1),k(-r+2),\ldots,k(m-1)) \\
&\times \prod_{l=-r+1}^{m-1} y_l^{k(l)}
\end{aligned} \tag{19}$$

is the generating function for the numbers $C(k(-r+1),k(-r+2),\ldots,k(m-1))$ of the restricted plane partitions with the sums $k(l)$ of the matrix elements (or parts of the partition) with the coordinates y_l, that is, positioned on the same (say l-th) diagonal in the matrix $\|a_{ij}\|$. This proves the first part of the theorem (eq. (4), (5)), where only for convenience all y_l are replaced by the products xy_l (assuming that x is complex variables and $|x| < 1$).

Now let $r \to \infty$, $m \to \infty$ and $y_l = x$ for all $-\infty \leqslant l \leqslant \infty$. Then the left hand side of equation (19) becomes

$$\prod_{p=1}^{\infty}(1-x^p)^{-p}. \tag{20}$$

But (20) is the generating function for all (unrestricted) plane partitions (see eq. (11.2.15) in [1]). Thus, the proof of the Proposition 2 is completed.

Consequently, the inverse function $F^{-1}: \|a_{ij}\| \to \|b_{ij}\|$ is uniquelly defined by F: $\|b_{ij}\| \to \|a_{ij}\|$. The explicit construction of F^{-1} is straightforward and is left to the reader (begin with the last nonzero element of the first column of the restricted plane partition $\|a_{ij}\|$).

The proof of the theorem is completed by noting that, when inserting b_{st}, in the product of y's in (18) the coordinates with nonnegative indices greater then $t-1$ do not appear and the coordinates with nonpositive indices less than $-s+1$ do not appear as well. Hence (6) and (7) follows.

4. Bender and Knuth [3] defined the one-to-one correspondence between plane partitions and the pairs of generalized Young tableaux of the same shape. The correspondence is constructed there by the use of Frobenius relation to each row (or column) of the plane partition. Let us note that there exists much more natural and simple way of associating plane partitions with the pairs of generalized Young tableaux. Namely, let us abserve that because of the constrains (1), (2) the parts of the plane partition on the leeding diagonal and above it define the Gelfand–Zetlin pattern (see, e.g., [11]). The same holds for the parts on the leading diagonal and below it. Thus, the restricted plane partition is ewuivalent to two Gelfand–Zetlin patterns with the same first row (the leading diagonal of the plane partition). Almost three dacades ago the present author tacitly assumed it as obvious (while constructing the bases of the representations of the unitary groups with the aid of Young operators) that Gelfand–Zetlin pattern id equivalent to the generalized Young tableau [12]. Let us formulate this equivalence in the terminology and notation of the present note. The shape of both the generalized Young tableaux in the correspondence is $\{a_{1i}, a_{22}, \ldots, a_{\min(r,m),\min(r,m)}\}$. The parts a_{ij} indices $(j-i)$ of coordinates of which are nonnegative integers, define the generalized Young tableau as follows: there are $(a_{ij} - a_{ij+1})$ numbers equal to $(m - j + i)$ in the i-th row of the generalized Young tableau. Analaogously, the parts a_{ij}, indices $(j-1)$ of coordinates of which are nonnegative integers, define the second generalized Young tableau of the same shape as follows: there are $(a_{ij} - a_{i+1j})$ numbers equal to $(r - i + j)$ in the j-th row of the generalized Young tableau.

Example. The plane partition $\|a_{ij}\|$ from (3) defines the following two generalized Young tableaux:

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1111222223   1112222334
22344        33344
4            4
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Note. From the theorem it follows that the number of appearances of the number p in the first generalized Young tableau is equal to the $(m-p+1)$-th column sum of the image $\|b_{ij}\|$ of the plane partition $\|a_{ij}\|$ in the bijection and is equal to the $(r-p+1)$-th row sum in $\|b_{ij}\|$ in the second generalized Young tableau.

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Plokščiųjų skaidinių ir sveikųjų neneigiamųjų matricų bijekcija

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Sukonstruota abipusiškai vienareikšmė atitiktis tarp plokštumos skaidinių su r eilučių ir m stulpelių ir sveikųjų neneigiamųjų skaičių matricų su r eilučių ir m stulpelių.

Rankraštis gautas
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