THE PROBLEM OF MULTIPLICITY OF SYMMETRICAL GROUP REPRESENTATION WHICH GENERALIZES THE PROBLEM OF PARTITION NUMBERING

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Vol. 26, No. 5, pp. 513-528, 1986

UDC 519.116-72:530.145+512.815.8

It is demonstrated that the number of whole-number nonnegative solutions of a Diophantine equation $\sum_{i=1}^{n(\beta)}\sum_{j=1}^{\beta_i} i\alpha_{ij}=k$ for variables of α_{ij} can be expressed in the form of a simple polynomial on k and [k/i] (i = 2, 3, ..., n(β)); only members such as $\alpha_{isr}k^S[k/i]^r$ appear in the polynomial. These polynomials are found for all partitions β with $m(\beta)=\sum_{i=1}^{n(\beta)} i\beta_i\leq 7$. A recurrent method of deduction is proposed for any value of the parameter m(β). The problem of determining the multiplicities of irreducible representations of a symmetrical group S_m in a representation of it by the action in a multitude of ordered partitions of the number k into no more than m parts and, as a special case, the problem of determining the number of the disordered partitions, are reduced to this problem. Formulas for calculating these numbers are cited for all m ≤ 7; in a certain sense they are simpler than those recently acquired by Colman. The use of the results for the problem of classifying the states of a multidimensional quantum-mechanical harmonic oscillator is examined.

1. The whole part of the rational number a will be designated as $[a]$ below. According to definition,

$$[-a]=-[a]-1 \quad \text{for} \quad a\neq 0 \quad \text{and} \quad [0]=0. \tag{1}$$

For a whole number k and a natural number i the following equality will be used as definition of the value $(k)_i$:

$$(k)_i=k-i[k/i], \tag{2}$$

i.e., $(k)_i$ is the least positive representative of a class of remainders in terms of the modulus i. Greek letters will be used to designate the partitions and their parts; where $\beta=\langle\beta_1,\beta_2,\ldots\rangle$, β_i is equal to the number of parts equal to i in the partition β ($\beta_i\geq 0$). Finite partitions are examined. There is a greater part of the n(β) partition of β; $m(\beta)=\sum_{i=1}^{n(\beta)} i\beta_i$, i.e., β is the partition of the number m(β). Derivatives of the function

$$\varphi_\beta(x)=\frac{1}{\prod_{i=1}^{n(\beta)}(1-x^i)^{\beta_i}}=\sum_{k=0}^{\infty} C_\beta(k)x^k \tag{3}$$

will be examined. From the expansion

$$\frac{1}{1-x}=\sum_{k=0}^{\infty} x^k \tag{4}$$

and from the rules of multiplication of exponential progressions it follows that $C_\beta(k)$

in (3) is equal to the number of solutions in whole, nonnegative numbers α_{ij} of the Diophantine (or undetermined) equation

$$\sum_{i=1}^{n(\beta)} \sum_{j=1}^{\beta_i} i\alpha_{ij} = k. \tag{5}$$

In a special case, when all $\beta_i = 1$ ($i = 1, 2, \ldots, m$), it is found in [1,2] that:

$$\frac{1}{\prod_{i=1}^{m} (1-x^i)} = \sum_{k=0}^{\infty} P_m^k x^k, \tag{6}$$

where P_m^k, as follows from (5) (with $n(\beta) = m$ and $\alpha_{i1} = \alpha_i$) is the number of disordered partitions of the numer k into parts which do not exceed m. From a representation of the partition of the number k in the form of a Young diagram (or a Ferraro graph) it is easy to see as in [2,3] that P_m^k may also be interpreted as the number of partitions into no more than m parts. When the number of partitions into equal to m parts is designated as $\bar{P}_m^k$, then the following equality will be the case

$$\bar{P}_m^k = P_m^{k-m}. \tag{7}$$

Recently Colman [4] found formulas for calculating $\bar{P}_m^k$ with $m = 2, 3, \ldots, 7$ which are much simpler in appearance than the previously known formulas. He also provided a method for sequential deduction of such formulas for any small value of m. (Without repeating the brief historical review of this question given by Colman and his further references, it is only noted that the first expressions for $\bar{P}_3^k$ and $\bar{P}_4^k$ were acquired by DeMorgan in 1843.) However, the Colman formulas contain trigonometric parts, while those of M. Hall ([1], p. 47) show that $\bar{P}_m^k$ may be expressed algebraically in the form of a polynomial on k of the m - 1 power with coefficients which are a function of the class of deductions in terms of the modulus m!, to which k belongs. Relying on information provided by this theorem of Hall, it would be possible to seek algebraic expressions for $\bar{P}_m^k$ in the form of polynomials of two variables k and $(k)_{m!}$. Here, in the general case the power of the polynomials with respect to the variable $(k)_{m!}$ is equal to m! - 1, since m! coefficients at k^s are a function of the value of the variable $(k)_{m!}$, they may be unambiguously expressed in the form of a polynomial with a power of m! - 1 of this variable using Lagrange's interpolation formula.

It is therefore surprising that there are simpler algebraic expressions not only for P_m^k, but also for $C_\beta(k)$ at any β (problem generalization), or, more precisely, $C_\beta(k)$ can be expressed in the form

$$C_\beta(k) = \sum_{i=1}^{n(\beta)} \sum_{r=1}^{i-1} M_{\beta ir}(k)(k)_i^r, \tag{8}$$

where $M_{\beta ir}(k)$ is the polynomial with a power no greater than $S_{\beta i}$ from the variable k,

$$S_{\beta i} = \Big(\sum_{i|j} \beta_j\Big) - 1. \tag{9}$$

Here $i|j$ means that j is divided into i without a remainder (in (8) it is assumed that $(k)_1^0 = 1$). It follows from (8) and (9) that, in particular, the coefficient at k^s in an algebraic expression for P_m^k is assigned by a substantially lower number of constants as compared with m! which is expected on the basis of the Hall theorem. Structural demonstration of the statement of (8) and (9) is one of the basic results of this work. Along with this, this work finds the polynomials $M_{\beta ir}(k)$ (and in the same way $C_\beta(k)$) for all β with $m(\beta) \leq 7$ and proposes a method for recurrent construction of expressions of $C_\beta(k)$ for any β and $m(\beta)$. Paragraph 7 of this work reduces the problem of determining the multiplicities of irreducible representations of the symmetrical group S_m in its representation in a multitude of ordered partitions of the number k into no more than m parts to this problem. A special case of the latter problem - numbering the disordered partitions - is examined in the last paragraph; in the formulas acquired using the method in this work there are no trigonometric parts which are present in the formulas acquired using the Colman method.

The concluding paragraph of this work examines one of the possible physical applications of the mathematical results acquired here. It is not the only problem of classification of the states of quantum mechanical many-particle systems. The problem of determining the multiplicities of states of configurations of equivalent particles in shell models with a quite large complete angular momentum (see [5-7]*) is reduced to the examined problem.

2. Demonstration of the statement of (8) is performed using a transfinite induction method. The multitude of partitions P is completely ordered by the introduction of a relation such as: $\beta<\alpha\ (\alpha,\beta\in P)$, when and only when for a certain i $\beta_i < \alpha_i$ and $\beta_s = \alpha_s$ for all s > i. It is noted that P contains all finite partitions; normally submultitudes of partitions with fixed $m(\gamma)$ are examined with orderings. The bases in the induction will be partitions such as $\langle 0, 0, \ldots, \gamma_i, 0, 0, \ldots\rangle \equiv \gamma(i)$. Since

$$\frac{1}{(1-x^i)^{\gamma_i}}=\sum_{k=0}^{\infty}\binom{k+\gamma_i-1}{\gamma_i-1}x^{ik}, \tag{10}$$

where $\binom{a}{b}$ is a binomial coefficient and, since

$$\frac{1}{(i-1)!}\sum_{s=1}^{i-1}\left(s-(k)_i\right)=\begin{cases}0, & \text{when } i\nmid k,\\ 1, & \text{when } i\mid k,\end{cases} \tag{11}$$

where $i\nmid k$ means that the remainder from dividing k by i is not equal to zero, then

$$C_{\gamma(i)}(k)=\frac{1}{(i-1)!\,(\gamma_i-1)!}\prod_{s=1}^{i-1}\left(s-(k)_i\right)\prod_{j=1}^{\gamma_i-1}\left(\frac{k}{i}+j\right). \tag{12}$$

In the same way it is tested that for base partitions of $\gamma(i)$ statement (8) is valid and an explicit form of the polynomials $M_{\gamma(i)\,\nu}(k)$ is found for these partitions.

The author offers the complete symmetrical functions [8-10] of h_l on variables of $x_1, x_2, \ldots,$ for examination

$$h_l=\sum_{i_1\leq i_2\leq\ldots\leq i_l}x_{i_1}x_{i_2}\ldots x_{i_l} \tag{13}$$

along with the symmetrical functions of exponential sums

$$S_l=\sum_i x_i^l \tag{14}$$

For partition of $\alpha\in P$ the following is assumed

$$h_\alpha=\prod_i h_i^{\alpha_i}, \tag{15}$$

$$S_\alpha=\prod_i S_i^{\alpha_i}. \tag{16}$$

The multitudes $\{h_\alpha\,|\,\alpha\in P\}$ and $\{S_\alpha\,|\,\alpha\in P\}$ are bases of a ring of symmetrical functions above a field of rational numbers [8-10] and, therefore, h_α may be unambiguously expressed in the form

$$h_\alpha=\sum_{\substack{\beta\\(m(\beta)=m(\alpha),\ \beta\leq\alpha)}}Q_{\alpha\beta}S_\beta. \tag{17}$$

*A clerical error is noted in [7]: 2π instead of π must be in the equality (6) under the sign of the threshold.

(work [10] provides tables of the coefficients of $Q_{\alpha\beta}$). In particular, for function (13), from [8-10], it is found that

$$h_i = \sum_{\substack{\gamma \\ (m(\gamma)=i)}} \frac{1}{\prod_j j^{\gamma_j} j!} S_\gamma \tag{18}$$

The equalities (15)-(18) assign the algorithm for finding the coefficients of $Q_{\alpha\beta}$. It is assumed that

$$x_i = x^{i-1}. \tag{19}$$

A line above the designation of the function will mean that substitution of (19) in it is complete. As a result of (3), (14), and (16)

$$\bar{S}_\beta = \varphi_\beta(x). \tag{20}$$

The following equality ([8], chapter 7 and [9], chapter 1, §3)*

$$\bar{h}_i = \frac{1}{\prod_{s=1}^{i} (1-x^s)}. \tag{21}$$

From (17), (20), and (21) it is found that

$$\varphi_{\tilde\alpha}(x) = \sum_{\substack{\beta \\ (m(\beta)=m(\alpha),\ \beta<\alpha)}} Q_{\alpha\beta}\,\varphi_\beta(x), \tag{22}$$

where $\tilde\alpha$ is a partition with

$$\tilde\alpha_i = \sum_{k \geqslant i} \alpha_k. \tag{23}$$

Since $\tilde\alpha > \beta$ follows from $\alpha \geq \beta$ (when $\alpha \neq \alpha(1)$), then for partitions $\tilde\alpha$, which are characterized by the fact that $\tilde\alpha_i \geqslant \tilde\alpha_{i+1}$ (as a result of (23)), the equality (22) makes it possible to find $C_{\tilde\alpha}(x)$ when $C_\beta(x)$ are known for the partitions $\tilde\alpha$, less than β. In the same way (8) is valid for $\tilde\alpha$, when it is valid for these β, since by taking a linear combination the property (8) is preserved (in the special case $\tilde\alpha = \alpha(1) = \langle a_1, 0, \ldots\rangle$, while $\varphi_{\alpha(1)}(x) = 1/(1-x)^{a_1} = \sum_{k=0}^{\infty} \binom{k+a_1-1}{a_1-1} x^k$ and the statement of (8) is also valid).

But for any partition γ it is assumed that

$$\bar{a}_i(\gamma) = \begin{cases} 0, & \text{when } \gamma_i \geqslant \max\{\gamma_s \mid s>i\}, \\ \max\{\gamma_s \mid s>i\} - \gamma_i, & \text{when } \gamma_i < \max\{\gamma_s \mid s>i\}. \end{cases} \tag{24}$$

Then $\tilde\alpha(\gamma)$ with $\tilde\alpha_i(\gamma) = \gamma_i + \bar{a}_i(\gamma)$ satisfies the inequality

$$\tilde\alpha_i(\gamma) \geqslant \tilde\alpha_{i+1}(\gamma). \tag{25}$$

and therefore, for $\tilde\alpha(\gamma)$ (22) and (23) with replacement of $\tilde\alpha$ for $\tilde\alpha(\gamma)$ and α for $\alpha(\gamma)$ ($\alpha_i(\gamma) = \tilde\alpha_i(\gamma) - \tilde\alpha_{i+1}(\gamma)$ may be used in accordance with (23)). It is found that

$$\begin{aligned} \varphi_\gamma(x) &= \prod_i (1-x^i)^{\bar{a}_i(\gamma)} \varphi_{\tilde\alpha(\gamma)}(x) = \\ &= \prod_i (1-x^i)^{\bar{a}_i(\gamma)} \sum_{\substack{\beta \\ (m(\beta)=m(\alpha(\gamma)),\ \beta<\alpha(\gamma))}} Q_{\alpha(\gamma)\beta}\,\varphi_\beta(x). \end{aligned} \tag{26}$$

Equality (26) is a basic one in our demonstration. In it

$$\gamma > \alpha(\gamma) \geqslant \beta. \tag{27}$$

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*It is noted that (21) along with (18) and (20) leads to the Kelly formula ([3], p. 215).

with the exception only of cases of $\gamma = \gamma(i)$ which, however, have already been examined as the basis cases for induction and for which statement (8) is valid (see (12)). Assume that

$$\varphi^{-1}_{\bar{\alpha}(\gamma)}(x) = \prod_i (1 - x^i)^{\bar{\alpha}_i(\gamma)} = \sum_{l=0}^{\sum_i \bar{\alpha}_i(\gamma)} D_{\bar{\alpha}(\gamma)}(l)\, x^l; \tag{28}$$

$\varphi^{-1}_{\bar{\alpha}(\gamma)}(x)$ is a polynomial (not an infinite progression). It is found from (3), (26), and (28) that

$$C_\gamma(k) = \sum_{\substack{\beta \\ (m(\beta)=m(\alpha(\gamma)),\ \beta<\alpha(\gamma))}} \sum_{l=0}^{\sum_i \bar{\alpha}_i(\gamma)} Q_{\alpha(\gamma)\beta}\, C_\beta(k-l)\, D_{\bar{\alpha}(\gamma)}(l). \tag{29}$$

$(k-l)_i^r$ is expressed in the form of a polynomial with an exponent of $i - 1$ of $(k)_i$. It is found that

$$\prod_{\substack{j=0 \\ (j\neq s)}}^{i-1} \frac{(j-(k)_i)}{j-s} = \delta_{(k)_i, s}, \tag{30}$$

where δ_{ab} is the Kronecker symbol. Since at $(l)_i \neq 0$

$$\begin{aligned}(k-l)_i^r = (k+i-(l)_i)_i^r = \sum_{s=0}^{(l)_i-1} (s+i-(l)_i)^r \delta_{(k)_i, s} + \\ + \sum_{s'=(l)_i+1}^{i-1} (s'-(l)_i)^r \delta_{(k)_i, s'},\end{aligned} \tag{31}$$

the substitution of (30) in (31) leads to the unknown expressions. Assuming that (8) is true for all $\beta < \gamma$, from (27), (30), and (31) it is found that $C_\gamma(k)$ also has the appearance of (8) (when $M_{\beta ir}(k)$ is a polynomial while $M_{\beta ir}(k - l)$ is also a polynomial).

In the same way statement (8) is structurally demonstrated, i.e., in the course of the demonstration an algorithm of recurrent finding of the polynomials $M_{\beta ir}(k)$ is constructed.

3. The authors demonstrate the validity of statement (9). Assume that $\beta^{(j)}$ is a partition with $\beta_j^{(j)} = \beta_j - 1$ and $\beta_s^{(j)} = \beta_s$ for $s \neq j$. It is found that

$$\varphi_\beta(x)(1-x^j) = \varphi_{\beta^{(j)}}(x), \tag{32}$$

from which it is found that

$$C_\beta(k) - C_\beta(k-j) = C_{\beta^{(j)}}(k). \tag{33}$$

Since $(k - j)_i = (k)_i$ at $i > 1$ when and only when $i|j$, then from (33) it follows that at $i > 1$, when and only when

$$M_{\beta ir}(k) - M_{\beta ir}(k-j) = M_{\beta^{(j)} ir}(k). \tag{34}$$

Therefore, for powers $S_{\beta ir}$ of the polynomials $M_{\beta ir}(k)$ $(i > 1)$ it is found that

$$S_{\beta ir} = S_{\beta^{(j)} ir}, \quad \text{if} \quad i \nmid j, \tag{35a}$$

and (as a result of (34)) either

$$M_{\beta^{(j)} ir}(k) = 0, \tag{35b}$$

or

$$S_{\beta ir} = S_{\beta^{(j)} ir} + 1, \quad \text{if} \quad i\, j. \tag{35c}$$

Assume that the statement (9) is valid for all partitions less than β (for basis partitions $\beta(i)$ (9) is valid as a result of (12)). Since $\beta^{(j)} < \beta$, it follows from (35) that it is also valid for β in the case of $i > 1$. For $i = 1$ the equality (34) in the general case is not satisfied, but $S_{\beta 1} < S_{\beta i}$ at $i > 1$ and $\beta \neq \beta(1)$ (when $\beta = \beta(1)$, then statement (9) is valid as a result of (10)-(12). Therefore, from (31) and (33) and the demonstrated validity of (9) for $i > 1$, it follows that the power $S_{\beta 10}$ of the polynomial $M_{\beta 10}(k)$ is one higher than the power of the polynomial $M_{\beta^{(1)}10}(k)$, and hence, sequentially removing a part of the partition β, it is found that

$$S_{\beta 10} = S_{\beta 1} = \sum_j \beta_j - 1$$

is precisely the case.

4. The authors consider an example. Assume that $\gamma = \langle a, 1, 2\rangle, a \geqslant 2$ (since there are no parts of partitions larger than 3, only the first three numbers are indicated in designations of the partitions). It is found that $\bar{\alpha}(\gamma) = \langle 0, 1, 0\rangle$, $\bar{\alpha}(\gamma) = \langle a, 2, 2\rangle$, $\alpha(\gamma) = \langle a-2, 0, 2\rangle$. From (17) and (18) it is found that

$$h_3 = 6^{-1}(S_1^3 + 3 S_1 S_2 + 2 S_3), \tag{36}$$

$$\begin{aligned} h_{\langle a-2, 0, 2\rangle} = 36^{-1}(S_1^{a+4} + 6 S_1^{a+2} S_2 + 4 S_1^{a+1} S_3 + \\ + 9 S_1^a S_2^2 + 12 S_1^{a-1} S_2 S_3 + 4 S_1^{a-2} S_3^2). \end{aligned} \tag{37}$$

Substitution of (19), the equality (20), and (26) produces

$$\begin{aligned} \varphi_{\langle a, 1, 2\rangle}(x) = (1 - x^2)\varphi_{\langle a, 2, 2\rangle}(x) = \frac{1-x^2}{36}\big(\varphi_{\langle a+4, 0, 0\rangle}(x) + \\ + 6\varphi_{\langle a+2, 1, 0\rangle}(x) + 4\varphi_{\langle a+1, 0, 1\rangle}(x) + 9\varphi_{\langle a, 2, 0\rangle}(x) + \\ + 12\varphi_{\langle a-1, 1, 1\rangle}(x) + 4\varphi_{\langle a-2, 0, 2\rangle}(x)\big). \end{aligned} \tag{38}$$

Assume that $a = 3$; for all partitions β the numbers 7 which appear in this case in the right side of the equality (38) and the expansion (3) are already known (see paragraph 6 of this work) and therefore, all that remains is to multiply by $(1 - x^2)$. This multiplication is examined for a member with $\beta = \langle 4, 0, 1\rangle$. It is found in paragraph 6 that

$$\begin{aligned} C_{\langle 4, 0, 1\rangle}(k) = 72^{-1}(k^4 + 14k^3 + 67k^2 + 126k + 72) + \\ + 2\cdot 9^{-1}(k)_2 - 9^{-1}(k)_3^2. \end{aligned} \tag{39}$$

The contribution of the $9^{-1}(1-x^2)\varphi_{\langle 4, 0, 1\rangle}(x)$ member to the expression for $C_{\langle 3, 1, 2\rangle}(k)$ is equal to

$$9^{-1}\big(C_{\langle 4, 0, 1\rangle}(k) - C_{\langle 4, 0, 1\rangle}(k-2)\big) = B. \tag{40}$$

Having placed the following expressions in (40) and (39) along with those found from (30) and (31)

$$(k-2)_2 = (k+1)_2 = 2^{-1}\big(-3(k)_3^2 + 5(k)_2 + 2\big),$$
$$(k-2)_3^2 = (k+1)_3^2 = 2^{-1}\big(-7(k)_3^2 + 13(k)_2 + 2\big),$$

it is found that

$$B = 162^{-1}(2k^3 + 15k^2 + 33k + 18) - 54^{-1}(k)_3^2 + 7\cdot 162^{-1}(k)_2. \tag{41}$$

In the three members of expression (38) the multiple $(1 - x^2)$ may be reduced and in this way their contributions may be expressed throgh $C_{\langle 3, 0, 0\rangle}(k)$, $C_{\langle 3, 1, 0\rangle}(k)$, $C_{\langle 2, 0, 1\rangle}(k)$. Summing the contributions of all six members, the following expression is found

$$\begin{aligned} C_{\langle 3, 1, 2\rangle}(k) = 12960^{-1}(6k^5 + 165k^4 + 1690k^3 + 7920k^2 + \\ + 16704k + 12960) - 32^{-1}(k)_2 - 54^{-1}(k+5)(k)_3^2 + \\ + 162^{-1}(3k + 13)(k)_2. \end{aligned} \tag{42}$$

5. The authors make the following observation. Since the type of formulas for $C_\beta(k)$ is already known, then relying on this available information it is possible to acquire any algorithms for finding $C_\beta(k)$, possibly at times simpler ones than those constructed in the course of the demonstration. In particular, it is possible to use the recurrent relation (33). There are also convenient relations of a special type used only for certain β. For instance,

$$\frac{1}{(1-x)^{m-4}(1-x^4)}=\frac{1}{2x}\left\{\frac{1}{(1-x)^{m-4}(1-x^3)}-\frac{1}{(1-x)^{m-4}(1-x^4)}\right\}, \qquad (43)$$

which may be used to sequentially find $C_\beta(k)$ for $\beta=\langle 1^{m-4}, 0, 0, 1\rangle$. It must be noted that when different algotihms are used, ambiguities may come up which come from the possibility of expressing $(k)_i$ through $(k)_j$ when $i|j$, for instance,

$$(k)_2=\frac{1}{3}\left(2(k)_4^3-9(k)_4^2+10(k)_4\right).$$

These ambiguities may be easily eliminated using the appropriate relations. Here, the conditions of $S_{\beta r}\leqslant S_{\beta i}$ may even be violated which, however, does not contradict the statement (9) since it only claims the possibility of fulfilling the condition.

6. This paragraph will present the expressions calculated by the authors for $C_\beta(k)$ for all β with $m(\beta) \le 7$ in a rising order of partitions β. The ambiguities will not be completely eliminated. It is only required that the powers of the polynomials $M_{\beta ir}(k)$ not exceed $S_{\beta i}$ for each $i \le n(\beta)$ which, as a result of the demonstrated statememt (9), is always possible.

$$C_{\langle 1\rangle}(k)=1,$$
$$C_{\langle 2\rangle}(k)=k+1,$$
$$C_{\langle 3\rangle}(k)=2^{-1}(k^2+3k+2),$$
$$C_{\langle 4\rangle}(k)=6^{-1}(k^3+6k^2+11k+6),$$
$$C_{\langle 5\rangle}(k)=24^{-1}(k^4+10k^3+35k^2+50k+24),$$
$$C_{\langle 6\rangle}(k)=120^{-1}(k^5+15k^4+85k^3+225k^2+274k+120),$$
$$C_{\langle 7\rangle}(k)=720^{-1}(k^6+21k^5+175k^4+735k^3+1624k^2+1764k+720),$$
$$C_{\langle 0,1\rangle}(k)=1-(k)_2,$$
$$C_{\langle 1,1\rangle}(k)=2^{-1}(k+2)-2^{-1}(k)_2,$$
$$C_{\langle 2,1\rangle}(k)=4^{-1}(k^2+4k+4)-4^{-1}(k)_2,$$
$$C_{\langle 3,1\rangle}(k)=24^{-1}(2k^3+15k^2+34k+24)-8^{-1}(k)_2,$$
$$C_{\langle 4,1\rangle}(k)=48^{-1}(k^4+12k^3+50k^2+84k+48)-16^{-1}(k)_2,$$
$$C_{\langle 5,1\rangle}(k)=480^{-1}(2k^5+35k^4+230k^3+700k^2+968k+480)-32^{-1}(k)_2,$$
$$C_{\langle 0,2\rangle}(k)=2^{-1}(k+2)-2^{-1}(k+2)(k)_2,$$
$$C_{\langle 1,2\rangle}(k)=8^{-1}(k^2+6k+8)-8^{-1}(2k+5)(k)_2,$$
$$C_{\langle 2,2\rangle}(k)=24^{-1}(k^3+9k^2+26k+24)-8^{-1}(k+3)(k)_2,$$
$$C_{\langle 3,2\rangle}(k)=96^{-1}(k^4+14k^3+68k^2+136k+96)-32^{-1}(2k+7)(k)_2,$$
$$C_{\langle 0,3\rangle}(k)=8^{-1}(k^2+6k+8)-8^{-1}(k^2+6k+8)(k)_2,$$
$$C_{\langle 1,3\rangle}(k)=48^{-1}(k^3+12k^2+44k+48)-16^{-1}(k^2+7k+11)(k)_2,$$
$$C_{\langle 0,0,1\rangle}(k)=1+2^{-1}(k)_3^2-3\cdot 2^{-1}(k)_3,$$
$$C_{\langle 1,0,1\rangle}(k)=3^{-1}(k+3)-3^{-1}(k)_3,$$
$$C_{\langle 2,0,1\rangle}(k)=6^{-1}(k^2+5k+6)-6^{-1}(k)_3^2+6^{-1}(k)_3,$$
$$C_{\langle 3,0,1\rangle}(k)=18^{-1}(k^3+9k^2+24k+18)-6^{-1}(k)_3^2+5\cdot 18^{-1}(k)_3,$$
$$C_{\langle 4,0,1\rangle}(k)=72^{-1}(k^4+14k^3+67k^2+126k+72)-9^{-1}(k)_3^2+2\cdot 9^{-1}(k)_3,$$
$$C_{\langle 0,1,1\rangle}(k)=6^{-1}(k+6)-2^{-1}(k)_2+2^{-1}(k)_3^2-7\cdot 6^{-1}(k)_3,$$
$$C_{\langle 1,1,1\rangle}(k)=12^{-1}(k^2+6k+12)-4^{-1}(k)_2+6^{-1}(k)_3^2-2^{-1}(k)_3,$$
$$C_{\langle 2,1,1\rangle}(k)=72^{-1}(2k^3+21k^2+66k+72)-8^{-1}(k)_2-9^{-1}(k)_3,$$
$$C_{\langle 0,2,1\rangle}(k)=24^{-1}(k^2+10k+24)-8^{-1}(2k+7)(k)_2+3^{-1}(k)_3^2-$$
$$-2\cdot 3^{-1}(k)_3,$$
$$C_{\langle 0,0,2\rangle}(k)=3^{-1}(k+3)+6^{-1}(k+3)(k)_3^2-2^{-1}(k+3)(k)_3,$$
$$C_{\langle 1,0,2\rangle}(k)=18^{-1}(k^2+9k+18)+18^{-1}(k)_3^2-18^{-1}(2k+9)(k)_3,$$
$$C_{\langle 0,0,0,1\rangle}(k)=1-6^{-1}(k)_4^3+(k)_4^2-11\cdot 6^{-1}(k)_4,$$
$$C_{\langle 1,0,0,1\rangle}(k)=4^{-1}(k+4)-4^{-1}(k)_4,$$

$$C_{\langle 2,0,0,1\rangle}(k)=8^{-1}(k^2+6k+8)-8^{-1}(k)_4^2+4^{-1}(k)_4,$$
$$C_{\langle 3,0,0,1\rangle}(k)=48^{-1}(2k^3+21k^2+64k+48)-24^{-1}(k)_4^3+16^{-1}(k)_4^2+$$
$$+6^{-1}(k)_4,$$
$$C_{\langle 0,1,0,1\rangle}(k)=4^{-1}(k+4)-4^{-1}k(k)_2-5\cdot 12^{-1}(k)_4^3+2(k)_4^2-$$
$$-31\cdot 12^{-1}(k)_4,$$
$$C_{\langle 1,1,0,1\rangle}(k)=16^{-1}(k^2+8k+16)-8^{-1}k(k)_2-5\cdot 24^{-1}(k)_4^3+$$
$$+15\cdot 16^{-1}(k)_4^2-7\cdot 6^{-1}(k)_4,$$
$$C_{\langle 0,0,1,1\rangle}(k)=12^{-1}(k+12)-3^{-1}(k)_3-6^{-1}(k)_4^3+(k)_4^2-19\cdot 12^{-1}(k)_4,$$
$$C_{\langle 0,0,0,0,1\rangle}(k)=1+24^{-1}(k)_5^4-5\cdot 12^{-1}(k)_5^3+35\cdot 24^{-1}(k)_5^2-25\cdot 12^{-1}(k)_5,$$
$$C_{\langle 1,0,0,0,1\rangle}(k)=5^{-1}(k+5)-5^{-1}(k)_5,$$
$$C_{\langle 2,0,0,0,1\rangle}(k)=10^{-1}(k^2+7k+10)-10^{-1}(k)_5^2+3\cdot 10^{-1}(k)_5,$$
$$C_{\langle 0,1,0,0,1\rangle}(k)=10^{-1}(k+10)+2^{-1}(k)_2+6^{-1}(k)_5^4-4\cdot 3^{-1}(k)_5^3+$$
$$+10\cdot 3^{-1}(k)_5^2-83\cdot 30^{-1}(k)_5,$$
$$C_{\langle 0,0,0,0,0,1\rangle}(k)=1-120^{-1}(k)_6^5+8^{-1}(k)_6^4-17\cdot 24^{-1}(k)_6^3+15\cdot 8^{-1}(k)_6^2-$$
$$-137\cdot 60^{-1}(k)_6,$$
$$C_{\langle 1,0,0,0,0,1\rangle}(k)=6^{-1}(k+6)-6^{-1}(k)_6,$$
$$C_{\langle 0,0,0,0,0,0,1\rangle}(k)=1+720^{-1}(k)_7^6-49\cdot 20^{-1}(k)_7^5+203\cdot 90^{-1}(k)_7^4-$$
$$-49\cdot 48^{-1}(k)_7^3+35\cdot 144^{-1}(k)_7^2-7\cdot 240^{-1}(k)_7.$$

7. The basic aspects of this paragraph are partially noted in [11]. Since irreducible representation of the symmetrical group S_m assigned by the λ partition is commonly designated through $[\lambda]$, it is hoped that the ambiguity in the understanding of the brackets will be eliminated by the context and by the use of Greek letters in one case and Latin letters in the other. The ordered partitions will be distinguished from the disordered in the same way. The ordered partition $y=(y_1, y_2, \ldots, y_m)$ of the number k into no more than m parts is solution in whole non-negative numbers of a simple Diophantine equation

$$\sum_{i=1}^{m} y_i=k; \tag{44}$$

y_i are the parts of the y partition. When $s\in S_m$ i.e., when s is transposition of the multitude $\{1, 2, \ldots, m\}=M$, and having used $s(i)\in M$ instead of $i\in M$, s shifts y_i into $y_{s(i)}$

$$y_i \mapsto y_{s(i)}. \tag{45}$$

This determines the effect of the S_m group on the multitude $Y_m(k)$ of the ordered partitions of the number k into no fewer than m parts. The corresponding transposed representation is designated as $R_m(k)$.

Assume that the disordered partition β assigns a class of equivalence of the group S_m with β_i equal to the number of cycles i long in a cyclic annotation of the transposition $s\in S_m$. Then the following is valid.

Proposition 1. $C_\beta(k)$ is equal to the nature of the class of equivalence β of the representation $R_m(k)$ of the group S_m by the action on the $Y_m(k)$ multiple.

In fact, the nature of the $s\in S_m$ element in the representation $R_m(k)$ is equal to the number of partitions $y\in Y_m(k)$, for which

$$y_{s(i)}=y_i \text{ for each } i\leq m. \tag{46}$$

Altogether (46) produces $\sum_{i=1}^{m}(i-1)\beta_i$ linearly independent equations. The authors examine (i - 1) equations which correspond to a single cycle $(p_1, p_2, \ldots, p_i)$ with a length of i (assuming that the cycle is the j-th in their specific numbering) of the transposition s

$$y_{p_1} = y_{p_2} = \dots = y_{p_t}. \tag{47}$$

General solution in whole, non-negative numbers of the system of Eqs. (47) is the following

$$y_{pi} = \alpha_{ij} \text{ for each } p_i, \tag{48}$$

where α_{ij} is any whole non-negative number. Considering the contributions such as (43) for each transposition cycle s, (5) is found from (44). Proposition 1 is demonstrated in the same way and, moreover, it is found that the following is valid.

Proposition 2. $\phi_\beta(x)$ is the derivative function for characters of the G_m group in a representation of it by its effect in the multitude of ordered partitions.

Assume that $\chi_\beta^{[\lambda]}$ is the irreducible property of the β class of equivalency with h_β elements. From proposition 1 and from the properties of orthogonality of the properties of the finite groups [8,12], it follows that

$$\frac{1}{m!}\sum_{\substack{\beta \\ (m(\beta)=m)}} h_\beta \chi_\beta^{[\lambda]} C_\beta(k) = c_m^{[\lambda]}(k) \tag{49}$$

is equal to the multiplicity of the irreducible representation $[\lambda]$ of the S_m group in the representation of $R_m(k)$. Since (see [8,9])

$$\{\lambda\} = \frac{1}{m!}\sum_{\substack{\beta \\ (m(\beta)=m)}} h_\beta \chi_\beta^{[\lambda]} S_\beta, \tag{50}$$

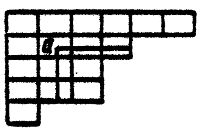
where $\{\lambda\}$ is the Schur function of the multitude of variables x_1 x_2, ..., it follows from (3), (20), (49), and (50) that the following is valid.

Proposition 3. The Schur function

$$\overline{\{\lambda\}} = \frac{1}{m!}\sum_{\substack{\beta \\ (m(\beta)=m)}} h_\beta \chi_\beta^{[\lambda]} \varphi_\beta(x) = \sum_{k=0}^{\infty} c_m^{[\lambda]}(k)\, x^k \tag{51}$$

is a derivative function for multiplicities of irreducible representations in the representation of the S_m group in the multitude $Y_m(k)$ of the ordered partitions into no more than m parts.

In the same way the authors came to the basic premise of this particular paragraph. Assume that $h(a)$ is the length of a hook (h00k) of cell a in a Young diagram which corresponds to the λ partition (see [9]). The concept of a hook and its length is illustrated by the following example:

λ —  . (52)

Here the cells which make up the hook of the a cell (the angular cell of the hook) are crossed out. The number of cells of the hook or its length is $h(a) = 5$. It is shown in [8], chapter 7 and in [9], chapter 1, §3 that

$$\overline{\{\lambda\}} = \prod_{a\in\lambda} \frac{x^{c(a)}}{(1-x^{h(a)})}, \tag{53}$$

where $c(a)$ is the number of cells above the a-th cell in the Young diagram which corresponds to the λ partition. In the example here $c(a) = 1$. A designation of $q(\lambda) = \sum_{a\in\lambda} c(a)$ is used. The sum of the numbers $h(a), a\in\lambda$ assigns a certain disordered partition β_λ from the multitude of P. β_λ makes up the submultitude $P' = \{\beta_\lambda \mid \lambda\in P\} \subset P$. From (3), (51), and (53) it is found that the following is true.

Proposition 4. The multiplicity $r_m^{[\lambda]}(k)$ of the irreducible representation $[\lambda]$ in the representation of the S_m group by the action in the multitude of ordered partitions $Y_m(k)$ is equal to $C_{\beta_\lambda}(k-q(\lambda))$.

As a result of proposition 4 the results of the previous paragraphs of this work with respect to $C_\beta(k)$ equally relate to the multiplicities $r_m^{[\lambda]}(k)$, producing, in particular, a method for deducing polynomial formulas for $r_m^{[\lambda]}(k)$.

8. The authors examine a narrow special case of the previous investigation which corresponds to a partition of $\lambda(m)$ which contains only a single part ($\lambda_m = 1$). Since this partition assigns the identical representation of the S_m group, all $\chi_\beta^{[\lambda(m)]} = 1$, and, since $h_\beta = m! \Big/ \prod_{i=1}^{m} i^{\beta_i}\beta_i!$, then it is found from (51), (53), and (6) that

$$\frac{1}{\prod_{i=1}^{m}(1-x^i)} = \sum_{(m(\beta)=m)}^{\beta} \frac{1}{\prod_{i=1}^{m} i^{\beta_i}\beta_i! \prod_{s=1}^{m}(1-x^s)^{\beta_s}} = \sum_{k=0}^{\infty} P_m^k x^k. \tag{54}$$

Here it is taken into consideration that for a single line Young diagram with a length of m the set of lengths of the hooks is equal to $\{1, 2, \ldots, m\}$, while $q(\lambda(m)) = 0$. Thus, the Kelly equality is encountered once again; this is the first equality in (54). On the left side of it is $\phi_{\beta'}(x)$ with $\beta'_1 = \beta'_2 = \cdots = \beta'_m = 1$ and $m(\beta') = m(m+1)/2$, and on the right side of it all $\phi_\beta(x)$ with $m(\beta) = m$ are summed. This equality is one of the recurrent relations constructed in paragraph 2 which make it possible to find formulas for coefficients in $\phi_{\beta'}(x)$ when they are already known for coefficients in $\phi_\beta(x)$ for all partitions β of the number m. The second equality in (54) is valid since it follows from (5) that $P_m^k = C_{\beta'}(k)$ is equal to the number of whole-number non-negative solutions of equation

$$\sum_{i=1}^{m} i\alpha_i = k \tag{55}$$

($\alpha_i = \alpha_{i1}$) is designated, i.e., to the number of disordered partitions of the number k into parts which do not exceed m (or partitions into no more than m parts). This may be interpreted differently in the following manner. As a result of proposition 3, P_m^k is the number of identical representations in the $R_m(k)$ representation. Each of the identical representations of the S_m group in the $Y_m(k)$ multitude mutually and unambiguously corresponds to a specific orbit of the transposing representation of $R_m(k)$ for $Y_m(k)$. In turn, each orbit is mutually and unambiguously assigned by the disordered partition which corresponds to the ordered partition of this orbit. In other words, P_m^k is also the number of the orbits of the representation $R_m(k)$ of the S_m group in the $Y_m(k)$ multitude.

These are the general tenets. The author now shifts to calculation of expressions for P_m^k. Since $C_\beta(k)$ for all β with $m(\beta) \leq 7$ are given in paragraph 6 of this work, then summing in terms of β in (54) should be performed to find formulas for P_m^k with $m \leq 7$. In each of the expressions acquired in this process there is a part as one of the components which is a linear combination of the form

$$O_m = \sum_{i=2}^{m} \sum_{r=1}^{i-1} c_{mir}(k)_i^r \tag{56}$$

(c_{mir} are the rational numbers). Through direct exhaustive search the author tested the fact that with the possible sets of values of $(k)_i$ the numerical values of O'_m of the value O_m for $m = 2, 3, \ldots, 7$ do not go beyond the limits of $1 < O'_m \leq 0$. Therefore, in formulas for P_m^k O_m can be discarded and the whole part of the remaining expression (enclosed in brackets) may be used. It is natural to assume that such a state of affairs is true for any m. The author currently has no demonstration of this assumption. Formulas are cited.

$$P_2^k=[2^{-1}k+1],$$
$$P_3^k=[12^{-1}k^2+2^{-1}k+1],$$
$$P_4^k=[144^{-1}k^3+5\cdot 48^{-1}k^2+2^{-1}k+1-16^{-1}k(k)_2],$$
$$P_5^k=[2880^{-1}k^4+96^{-1}k^3+31\cdot 288^{-1}k^2+11\cdot 24^{-1}k+1-32^{-1}k(k)_2],$$
$$\begin{aligned}P_6^k=[&86\,400^{-1}k^5+7\cdot 11\,520^{-1}k^4+77\cdot 6480^{-1}k^3+\\&+31\cdot 288^{-1}k^2+167\cdot 360^{-1}k+1-384^{-1}k^2(k)_2-\\&-7\cdot 128^{-1}k(k)_2+108^{-1}k(k)_3^2-36^{-1}k(k)_3],\end{aligned}$$
$$\begin{aligned}P_7^k=[&3\,628\,800^{-1}k^6+7\cdot 302\,400^{-1}k^5+553\cdot 725\,760^{-1}k^4+\\&+161\cdot 12\,960^{-1}k^3+251\cdot 2400^{-1}k^2+157\cdot 360^{-1}k+1-\\&-768^{-1}k^2(k)_2-7\cdot 192^{-1}k(k)_2-162^{-1}k(k)_3].\end{aligned}$$

When the formula is used for calculating the numerical value of P_m^k, then it is convenient to use the above cited formulas. But when a formula is required for performing further algebraic calculations, then the residual member of O_m must be known; in this case the brackets in the above cited formulas should be dropped and O_m added. These are cited below:

$$O_2=-2^{-1}(k)_2,$$
$$O_3=-4^{-1}(k)_2+6^{-1}(k)_3^2-2^{-1}(k)_3,$$
$$O_4=-9^{-1}(k)_3-6^{-1}(k)_4^3+13\cdot 16^{-1}(k)_4^2-13\cdot 12^{-1}(k)_4,$$
$$\begin{aligned}O_5=&\,18^{-1}(k)_3^2-6^{-1}(k)_3-11\cdot 96^{-1}(k)_4^3+33\cdot 64^{-1}(k)_4^2-\\&-61\cdot 96^{-1}(k)_4+120^{-1}(k)_5^4-12^{-1}(k)_5^3+7\cdot 24^{-1}(k)_5^2-5\cdot 12^{-1}(k)_5,\end{aligned}$$
$$\begin{aligned}O_6=&-127\cdot 1152^{-1}(k)_4^3+127\cdot 256^{-1}(k)_4^2-671\cdot 1152^{-1}(k)_4-\\&-25^{-1}(k)_5+135^{-1}(k)_6^5-12^{-1}(k)_6^4+95\cdot 324^{-1}(k)_6^3-\\&-5\cdot 18^{-1}(k)_6^2-19\cdot 90^{-1}(k)_6,\end{aligned}$$
$$\begin{aligned}O_7=&-137\cdot 1152^{-1}(k)_4^3+141\cdot 256^{-1}(k)_4^2-757\cdot 1152^{-1}(k)_4+\\&+60^{-1}(k)_5^4-2\cdot 15^{-1}(k)_5^3+97\cdot 300^{-1}(k)_5^2-37\cdot 150^{-1}(k)_5+\\&+57\cdot 6480^{-1}(k)_6^5-35\cdot 324^{-1}(k)_6^4+583\cdot 1296^{-1}(k)_6^3-115\cdot 162^{-1}(k)_6^2+\\&+139\cdot 540^{-1}(k)_6+5040^{-1}(k)_7^6-240^{-1}(k)_7^5+5\cdot 144^{-1}(k)_7^4-\\&-7\cdot 48^{-1}(k)_7^3+87\cdot 270^{-1}(k)_7^2-7\cdot 20^{-1}(k)_7.\end{aligned}$$

9. The acquired results may be used for classifying the states of many-particle quantum mechanical systems. Assume that H_0 is the Hamiltonian of non-interacting m particles with a common single-particle potential of a harmonic oscillator ([13], p. 345). H_0 is invariant with respect to transformations of the unitary group U_{3m}; its characteristic functions, which correspond to the number of quanta, are the basis of a symmetrical representation of this group assigned by a single-line Young diagram[n]. With a narrowing of the U_{3m} group to a subgroup of U_3 by U_m this representation is broken down into representations of the subgroup assigned by three-line Young diagrams [ν]. When constructing the wave function which has irreducible transformational properties relative to the group of transformations of the spatial coordinates, the group U_m is narrowed to the symmetrical subgroup S_m. It is in this step that the problem of determining the multiplicity $C_{\mu\nu}$ of the appearance of the representation [ν] of the S_m group in the representation [ν] of the U_m group comes up. With specific assumptions relative to the perturbating interaction of particles (for instance, in a supermultiplet model of an atomic nucleus [14]), $C_{\nu\mu}$ is also the degree of deformation or the statistical weight of the energy level. It was found in work [15]* that in the designations of this work

*Expression (18) and the left side of equality (19) of work [15] must be replaced with $\prod_{j=1}^{q}\prod_{l_1,\dots,l_m=0}^{\infty}\left(1-x_j\prod_{i=1}^{m}y_i^{l_i}\right)^{-1}$, while $\prod_{s=1}^{m}(1-y_s)\prod_{j=1}^{q}\prod_{l_1,\dots,l_m=0}^{\infty}\left(1-x_j\prod_{i=1}^{m}y_i^{l_i}\right)^{-1}$ must be in the left side of (24).

$$\frac{1}{m!} \sum_{\substack{\beta \\ (m(\beta)=m)}} \prod_{i=1}^{p} \varphi(x_i)\, h_\beta \chi_\beta^{[\mu]} = \sum_{\nu} c_{\mu\nu} \{\nu\}, \tag{57}$$

where p may be assumed to be equal to the number of lines in the Young diagram $[\nu]$. Expanding the expression on the left side of equality (57) in terms of the Schur function $\{\nu\}$ (see [8,9]), in the case of $p = 3$, it is found that

$$\begin{aligned} C_{\mu\nu} = \frac{1}{m!} \sum_{\substack{\beta \\ (m(\beta)=m)}} & h_\beta \chi_\beta^{[\mu]} \{ C_\beta({}_1\nu)\, C_\beta({}_2\nu)\, C_\beta({}_3\nu) - C_\beta({}_1\nu+1) \times \\ & \times C_\beta({}_2\nu-1)\, C_\beta({}_3\nu) - C_\beta({}_1\nu)\, C_\beta({}_2\nu+1)\, C_\beta({}_3\nu-1) - \\ & - C_\beta({}_1\nu+2)\, C_\beta({}_2\nu)\, C_\beta({}_3\nu-2) + C_\beta({}_1\nu+2)\, C_\beta({}_2\nu-1)\, C_\beta({}_3\nu-1) + \\ & + C_\beta({}_1\nu+1)\, C_\beta({}_2\nu+1)\, C_\beta({}_3\nu-2)\}, \end{aligned} \tag{58}$$

where ${}_i\nu$ is the length of the i-th line of the Young diagram $[\nu]$. Substituting in (58) the polynomial expressions for $C_\beta(k)$ examined in the previous paragraphs of this work, explicit formulas are found for calculating the multiplicities of $C_{\mu\nu}$. When ${}_2\nu = {}_3\nu = 0$ the left side of the equality (57) is greatly simplified (see equality (20) in [15]) and it produces

$$x^{q(\mu)} \prod_{a \in \mu} \frac{1}{1 - x^{h(a)}} = \sum_{{}_1\nu=0}^{\infty} C_{\mu\, {}_1\nu}\, x^{{}_1\nu} \tag{59}$$

in the designations of paragraph 7 of this work, from which

$$C_{\mu\, {}_1\nu} \equiv C_{\mu k} = C_{\beta_\mu}(k - q(\mu)) \tag{60}$$

is found for the symmetrical representation $[{}_1\nu] \equiv [k]$ of the U_m group.

In conclusion the author examines the case of a translationally invariant Hamiltonian of the harmonic oscillator $\bar{H}_0$ ([13], p. 380 and [14]). In this case the U_{m-1} group is narrowed to the S_m subgroup. For multiplicities $V_{\mu\nu}$ of irreducible representations $[\mu]$ of the S_m group in irreducible representations $[\nu]$ of the U_{m-1} group, an expression is found from equality (24) of work [15] with consideration of the correction made here in footnote which differs from the right side of (58) by replacement of all $C_\beta(k)$ with $C_{\beta^{(\mu)}}(k)$ ($\beta^{(\mu)}$ identified in paragraph 3; however, a negative parameter of $\beta_1^{(\mu)} = -1$ at $\beta_1 = 0$ is allowed in this case. For a symmetrical representation $[{}_1\nu] \equiv [k]$ of the U_{m-1} group, the following is found

$$x^{q(\mu)} \frac{1 - x}{\prod_{a \in \mu} (1 - x^{h(a)})} = \sum_{k=0}^{\infty} V_{\mu k}\, x^{k}, \tag{61}$$

$$V_{\mu k} = C_{\beta_\mu}(k - q(\mu)) - C_{\beta_\mu}(k - q(\mu) - 1). \tag{62}$$

instead of (59) and (60).

Formulas for $C_{\mu\nu}$ and $V_{\mu\nu}$, acquired using the method formulated here will be cited in a separate publication at greater values of the m parameter.

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5 December 1985

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